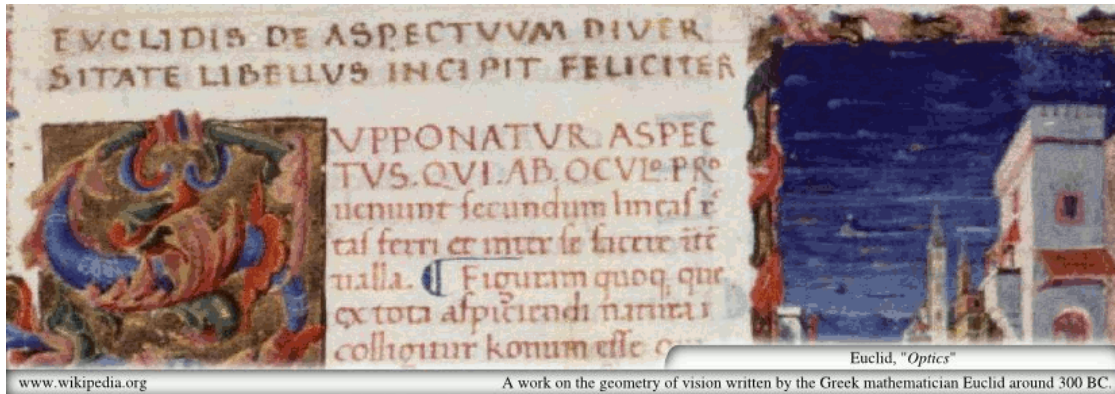


## Academic Publications International Journal of Pure and Applied Mathematics

ISSN 1311-8080 (printed version)  
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SCOPUS Source Normalized Impact per Paper (SNIP): 0.252 (2013)

SCOPUS SCImago Journal Rank (SJR): 0.605 (2013)

Google Scholar: H-Index: 20↑; G-Index: 31↑; Cites/paper: 2.80↑; Cites/year: 233.17↑



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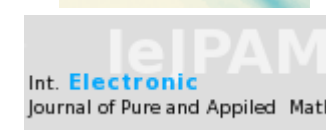
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## SOME GRAPHS IN $C_f2$ BASED ON $f$ -COLORING

Adiwijaya<sup>1 §</sup>, A.N.M. Salman<sup>2</sup>, O. Serra<sup>3</sup>, D. Suprijanto<sup>4</sup>, E.T. Baskoro<sup>5</sup>

<sup>1</sup>Telkom University

Jl. Telekomunikasi 1, Bandung 40257, INDONESIA

<sup>2,4,5</sup>Bandung Institute of Technology

Jl. Ganesa 10, Bandung 40132, INDONESIA

<sup>3</sup>Universitat Politècnica de Catalunya (UPC)

C/Jordi Girona 1-3, E-08034 Barcelona, SPAIN

**Abstract:** Let  $G = (V, E)$  be a graph and  $f : V \rightarrow Z^+$  a positive integer be a function. An  $f$ -coloring of  $G$  is a coloring of the edges such that every vertex  $v \in V$  is incident to at most  $f(v)$  edges of the same color. The minimum number of colors of an  $f$ -coloring of  $G$  is the  $f$ -chromatic index  $\chi'_f(G)$  of  $G$ . Based on the  $f$ -chromatic index, a graph  $G$  can be either in class  $C_f1$ , if  $\chi'_f(G) = \Delta_f(G)$ , or in class  $C_f2$ , if  $\chi'_f(G) = \Delta_f(G) + 1$ , where  $\Delta_f(G) = \max_{x \in V} \lceil d(x)/f(x) \rceil$ . In this paper, we give some sufficient conditions for a graph to be in  $C_f2$ . One of the results is a generalization of a theorem by Zhang *et al.* (2008). Moreover, we show that, when  $f$  is constant and a divisor of  $(n - 1)$ , a maximal subgraph of the complete graph  $K_n$  which is in class  $C_f1$  has precisely  $\binom{n}{2} - \Delta_f(K_n)/2$  edges.

**AMS Subject Classification:** 05C15

**Key Words:** edge coloring,  $f$ -coloring,  $f$ -chromatic index

### 1. Introduction

Let  $G = (V, E)$  be a finite and simple graph and let  $f$  be a function from  $V$  to a positive integer set. An  $f$ -coloring of  $G$  is a coloring of the edges such that

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Received: January 20, 2015

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<sup>§</sup>Correspondence author

each vertex  $v$  has at most  $f(v)$  edges colored with a same color. The minimum number of colors needed to define an  $f$ -coloring of  $G$  is called the  $f$ -chromatic index of  $G$ , denoted by  $\chi'_f(G)$ . In case  $f(v) = 1$  for every  $v \in V$ , an  $f$ -coloring is just a proper edge-coloring of the graph.

The  $f$ -colorings arise in many applications, including network design, scheduling problems and the file transfer problem in computer networks. For instance, the file transfer problem in a computer network is modeled as follows. Each computer is represented by a vertex and the file transfer process between two computers is represented by an edge. Each computer  $v$  has a limited number  $f(v)$  of communication ports. If we assume that the transfer time is constant for every file, we can use an  $f$ -coloring to manage transferring all files with minimum time. Being a generalization of the proper edge-coloring, which was shown to be an NP-complete problem by Holyer [5], the  $f$ -coloring problem is also NP-complete.

Let  $d(v)$  denote the degree of  $v \in V$ . By extending the well-known theorem of Vizing (1965) to  $f$ -colorings, Hakimi and Kariv [4] showed that

$$\Delta_f(G) \leq \chi'_f(G) \leq \Delta_f(G) + 1, \quad (1)$$

where

$$\Delta_f(G) = \sum_{i=1}^n \left\lceil \frac{d(v)}{f(v)} \right\rceil. \quad (2)$$

According to (1),  $G$  is said to be in *class-1*, denoted by  $C_f1$ , if  $\chi'_f(G) = \Delta_f(G)$ ; otherwise  $G$  is in *class-2*, denoted by  $C_f2$ . As for the corresponding classification for proper edge-colorings, graphs in class  $C_f2$  are much less common than the ones in class  $C_f1$ .

Hakimi and Kariv [4] showed that any bipartite graph is in  $C_f1$ . Moreover, for any graph, they showed that, if  $f(v)$  is even for each  $v \in G$ , then  $G$  is in  $C_f1$ . Zhang and Liu [6] gave the following classification of complete graphs based on  $f$ -colorings.

**Theorem 1** (Zhang and Liu [6]). *If  $k$  and  $n$  are odd integers with  $n \geq 3$ ,  $f(v) = k$  for all  $v \in V(K_n)$ , and  $k$  divides  $n - 1$ , then the complete graph  $K_n$  is in  $C_f2$ . Otherwise,  $K_n$  is in  $C_f1$ .*

Zhang, Wang and Liu [7] gave the following sufficient condition for a regular graph to be in  $C_f2$ .

**Theorem 2** (Zhang, Wang and Liu [7]). *Let  $n \geq 1$ . Let  $G$  be a  $\Delta$ -regular*

graph of order  $2n + 1$  and let  $k$  be an odd positive integer such that  $k$  divides  $\Delta$ . If  $f(v) = k$  for each  $v \in V$ , then  $G \in C_f2$ .

In [1] we gave a classification of some graphs containing wheels, namely the corona product of either the complement of a complete graph, or a path, or a star with a cycle, based on  $f$ -coloring. As well known, the corona product is not commutative. Hence, in [2] we provide a characterization of the corona product of a cycle with some graphs based  $f$ -chromatic index.

In this paper we give a sufficient condition for a graph to be in  $C_f2$  (Theorem 3). We also generalize Theorem 2 of Zhang, Wang and Liu [7], by dropping the condition of regularity of a graph (see Theorem 4). Finally, in Theorem 5, we provide an edge-reduction of a complete graph which is in  $C_f2$  in order to get a maximal subgraph of  $K_n$  which is in  $C_f1$ .

## 2. Main Results

Theorem 3 below provides a sufficient condition for a graph containing odd number of edges to be in  $C_f2$ . It can be seen as an extension to  $f$ -colorings of the fact that odd cycles belong to class 2 under ordinary proper edge-colorings.

**Theorem 3.** *Let  $G = (V, E)$ . Let  $f : V \rightarrow$  a positive integer be a function defined as  $f(v) = d(v)/k$  for each  $v \in V$ , where  $k$  is even. If  $|E|$  is odd, then  $G \in C_f2$ .*

*Proof.* Suppose on the contrary that  $G \in C_f1$ . Let  $C$  be an  $f$ -coloring which uses  $\Delta_f(G) = k$  colors.

We have

$$\sum_{v \in V} f(v) = \frac{1}{k} \sum_{v \in V} d(v) = \frac{2}{k} |E|.$$

Since  $|E|$  is odd and  $k$  is even, we have  $\sum_{v \in V} f(v) \equiv 1 \pmod{2}$ . Since the  $f$ -coloring  $C$  uses  $k$ -colors, each vertex  $v$  is incident with  $d(v)/k$  colors. But this means that each color class contains  $(1/2) \sum_{v \in V} d(v)/k = (1/2) \sum_{v \in V} f(v)$  edges, which is impossible since  $\sum_{v \in V} f(v)$  is odd.  $\square$

Figure 1 shows an example of a graph  $G \in C_f2$  where  $f(v) = d(v)/2$ . The fact that  $G \in C_f2$  follows from Theorem 3. However, this fact is not implied by Theorem 2 of Zhang, Wang and Liu [7], which applies only to regular graphs. However, this fact is not implied by Theorem 2 of Zhang, Wang and

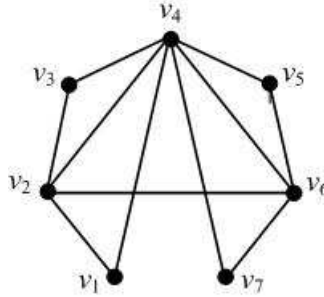


Figure 1: A non-regular graph that is in  $C_f2$

Liu [7], which applies only to regular graphs. In Theorem 4, we give a sufficient condition for a graph to be in  $C_f2$  which generalizes Theorem 2 since it does not require the graph to be regular and  $f(v)$  is not necessarily odd for each vertex.

**Theorem 4.** *Let  $G = (V, E)$  be a graph and let  $g = \gcd(d(v) : v \in V)$ . Let  $f : V \rightarrow$  a positive integer be defined as  $f(v) = d(v)/k$  for each  $v \in V$ , where  $k$  divides  $g$ . If  $V_1 = \{v \in V : f(v) \equiv 1 \pmod{2}\}$  and  $|V_1| \equiv 1 \pmod{2}$ , then  $G \in C_f2$ .*

*Proof.* Suppose on the contrary that  $G \in C_f1$ . Let  $C$  be an  $f$ -coloring that uses  $\Delta_f(G) = k$  colors. Let,  $V_2 = V \setminus V_1$ . Since the  $f$ -coloring  $C$  uses  $k$ -colors, then, by the definition of  $f$ , each vertex  $v$  is incident with  $d(v)/k$  colors. Therefore each color class  $E_i$  must contain

$$|E_i| = \frac{1}{2} \sum_{v \in V} d(v)/k = \frac{1}{2} \sum_{v \in V} f(v) = \frac{1}{2} \left( \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v) \right)$$

edges. By the definition of  $V_2$  we have  $\sum_{v \in V_2} f(v) \equiv 0 \pmod{2}$ . Since  $|V_1|$  is odd then, again by the definition of the set  $V_1$ , we have  $\sum_{v \in V_1} f(v) \equiv 1 \pmod{2}$  and therefore  $\sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v)$  is an odd number. But then  $|E_i|$  is not an integer, a contradiction.  $\square$

According to the classification of Zhang *et al.* [6] as we rewrite in Theorem 1, the only functions  $f$  for which complete graphs with order odd  $n$  are in class  $C_f2$  are the ones defined by  $f(v) = k$  for some odd divisor  $k$  of  $n - 1$ . A natural question is how many edges have to be removed from  $K_n$  for such an  $f$  so that



the remaining graph moves from  $C_f2$  to  $C_f1$ . The following theorem answers the question.

**Theorem 5.** *Let  $k$  and  $n$  be odd integers with  $n \geq 3$  and  $k$  divides  $(n-1)$ , and let  $f(v) = k$  for all  $v \in V(K_n)$ . If  $H$  is a maximal subgraph of  $K_n$  in  $C_f1$  then*

$$|E(H)| = |E(K_n)| - (n-1)/2k.$$

*Proof.* From Theorem 1 we know that  $K_n \in C_f2$ . We divide the proof into two cases as follows.

**Case 1.**  $f(v) = 1$  for all  $v \in V(K_n)$ .

In this case an  $f$ -coloring is a proper edge-coloring. Since  $H$  is in  $C_f1$  it can be  $f$ -colored with  $(n-1)$  colors. This means that  $H$  can be decomposed into  $(n-1)$  edge-disjoint matchings. Since each matching has at most  $(n-1)/2$  edges, we have  $|E(H)| \leq (n-1)^2/2 = |E(K_n)| - (n-1)/2$ . On the other hand, it is well-known that the complete graph  $K_n$  where  $n$  is odd, admits a decomposition into  $n$  edge-disjoint matchings with  $(n-1)/2$  edges each. By removing one of the matchings we obtain a subgraph  $H$  which can be properly edge colored with  $(n-1)$  colors and has  $|E(K_n)| - (n-1)/2$  edges.

**Case 2.** We have  $f(v) = 2r + 1$  for all  $v \in V$  and some positive integer  $r$ .

Let  $\Delta = \Delta_f = (n-1)/(2r+1)$  and let  $H$  be a subgraph of  $K_n$  obtained by removing a matching  $M$  of size  $\Delta/2$ . We will show that  $H \in C_f1$ . This will show that there is a subgraph of  $K_n$  with  $\binom{n}{2} - (n-1)/(4r+2)$  edges belonging to the class  $C_f1$ .

Let  $W = \{w_1, \dots, w_\Delta\}$  be the set vertices incident to the edges in  $M$  and let  $U = \{u_1, \dots, u_{n-\Delta}\}$  be the set remaining vertices of  $K_n$ . Consider the complete subgraph  $K_{n-\Delta}$  of  $K_n$  induced by  $U$ . It is well-known that the edge set of a complete graph of odd order can be decomposed into edge-disjoint hamiltonian cycles (Lucas Theorem). Consider such a decomposition of  $K_{n-\Delta}$  into  $(n-\Delta-1)/2 = r(n-1)/(2r+1)$  edge-disjoint hamiltonian cycles. Partition this set of hamiltonian cycles into  $(n-1)/(2r+1)$  classes where the cardinality of every class is  $r$ . Give a different color to each class. This results an  $f$ -coloring of the edges of  $K_{n-\Delta}$  with  $\Delta$  colors in such a way that each vertex is incident with  $2r$  edges of each color.

We next partition the vertices of  $U \setminus \{u_{n-\Delta}\}$  into  $\Delta$  subsets  $U_1, \dots, U_\Delta$  with  $(n-\Delta-1)/\Delta = 2r$  elements each. Color the edges joining  $w_i$  with  $U_j$  with

color  $(i + j) \pmod{\Delta}$  (in case  $(i + j) \equiv 0 \pmod{\Delta}$ , we color the edges by color  $\Delta$ ). At this point every vertex in  $U \setminus \{u_{n-\Delta}\}$  is incident with precisely  $2r + 1$  edges of each color, and every vertex in  $W$  is incident with  $2r$  edges of each color. We next color the edge  $u_{n-\Delta}w_i$  with color  $i$ , so that  $u_{n-\Delta}$  is incident to  $2r + 1$  edges of each color and vertex  $w_i$  is incident with  $2r + 1$  edges of color  $i$  and  $2r$  edges of color  $j$  for each  $j \neq i$ . It remains to show that we can color the edges of the subgraph  $H[W]$  of  $H$  induced by  $W$  in such a way that vertex  $w_i$  is not incident with color  $i$  for each  $i$ .

Consider the complete graph  $K_{\Delta+1}$ . It admits a decomposition into  $\Delta + 1$  matchings of size  $\Delta/2$  each. Consider the proper edge coloring of  $K_{\Delta+1}$  given by this decomposition. By removing one color class, we obtain a graph with a proper edge-coloring where one vertex incident to  $\Delta$  colors and the  $\Delta$  remaining vertices each adjacent to  $\Delta - 1$  colors. The subgraph induced by these  $\Delta$  vertices is isomorphic to  $H[W]$ , and, out of the  $\Delta$  colors, each vertex misses one. Moreover, no two vertices have the same missing color. By appropriately renaming the colors, one may require that this missing color at vertex  $w_i$  is color  $i$ . By using this coloring in the graph  $H[W]$  above we complete a coloring of  $H$  with  $\Delta$  colors such that every vertex is incident with precisely  $2r + 1$  edges of each color. Therefore  $H \in C_f1$ .

To complete the proof we need to show that every subgraph  $H \subset K_n$  that belongs to  $C_f1$  has at most  $\binom{n}{2} - (n - 1)/2k$  edges. But this follows from the fact that in an  $f$ -coloring with  $\Delta$  colors of any graph  $H$  with  $\binom{n}{2} - (n - 1)/2k$  edges, every vertex must be incident with precisely  $k$  edges of each color, so that the addition of any edges increases the frequency of one color above  $k$ . This completes the proof.  $\square$

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