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A characterization of the corona product of a cycle with some graphs based on its f-chromatic index[1]

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Abstract. An f-coloring of graph $G(V,E)$ is a generalized edge-coloring such that every vertex v in V has at most $f(v)$ edges colored with a same color. There are many applications of the f-coloring, for instance we can use it in order to solve some scheduling problems and some network design problems. The minimum number of colors needed to f-color G is called an the f-chromatic index of G, denoted by $\chi'_f(G)$. Any graph G has f-chromatic index equal to $\Delta_f(G)$ or $\Delta_f(G)+1$, where $\Delta_f(G) = max_{v \in V}\{\lceil d(v)/f(v) \rceil\}$. G is called in the *class-1*, denoted by $G \in C_f 1$, if $\chi'_f(G) = \Delta_f(G)$; otherwise G is called in the *class-2*, denoted by $G \in C_f 2$. In this paper, we show that the corona product of C_n with S_m is in $C_f 1$. Besides that, we characterize the corona product of C_n with either W_n or K_n based on f-coloring.

Keywords: corona product, f-chromatic index, f-coloring, graph
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INTRODUCTION

Throughout this paper, $G(V,E)$ is a simple and finite graph with the vertex set $V(G)$ and the edge set $E(G)$. We refer to [6] for undefined terms. Let f be a function from $V(G)$ to a subset of the positive integers. An *f-coloring* of G is a coloring of edges such that each vertex v has at most $f(v)$ edges colored with a same color. The minimum number of colors needed in the f-coloring of G is called an *f-chromatic index* of G, denoted by $\chi'_f(G)$.

A problem in the f-coloring is how to determine $\chi'_f(G)$ of a given graph G. It arises in many applications, including the network design problem, the scheduling problem, and the file transfer problem in a computer network [4]. The file transfer problem in a computer network is modeled as follows. Each computer is represented by a vertex and every two computers in the file transfer process is represented by an edge. Each computer v has a limit number $f(v)$ of communication ports. If we assume that the transfer time is constant for every file, we can use an f-coloring to manage transferring all files along the minimum time needed.

Let $d(v)$ denote the degree of $v \in V$. By extending the well–known theorem of Vizing (1965) to f-colorings, Hakimi and Kariv [5] showed that

$$\Delta_f(G) \leq \chi'_f(G) \leq \Delta_f(G)+1,$$

where

$$\Delta_f(G) = \max_{v \in V(G)} \left\{ \left\lceil \frac{d(v)}{f(v)} \right\rceil \right\}.$$

According to [5], G is called in the *class-1*, denoted by $G \in C_f 1$, if $\chi'_f(G) = \Delta_f(G)$; otherwise G is called in the *class-2*, denoted by $G \in C_f 2$. Holyer [7] proved that the edge-coloring problem is an NP-complete. It is reduced from the 3SAT problem. Consequently, the f-coloring problem is an NP-complete problem.

Hakimi and Kariv [5] showed that any bipartite graph is in $C_f 1$. Moreover, for any graph, they showed that, if $f(v)$ is even for each $v \in G$, then G is in $C_f 1$. Yu et al. [8] gave sufficient conditions for fans to be in $C_f 1$. Zhang and Liu [10] found the f-chromatic index for complete graphs and gave a classification of complete graphs on f-coloring. Let,

$$V_0^*(G) = \left\{ v : \frac{d(v)}{f(v)} = \Delta_f(G), v \in V(G) \right\},$$

and

$$V^*(G) = \left\{ v : \left\lceil \frac{d(v)}{f(v)} \right\rceil = \Delta_f(G), v \in V(G) \right\}.$$

Zhang and Liu in [9] gave some sufficient conditions for a graph to be in $C_f 1$ as follows.

Theorem 1. *Let G be a graph. If the subgraph induced by $V_0{}^*(G)$ is forest, then $G \in C_f 1$.*

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Theorem 2. *Let G be a graph. If $d(v^*)$ is not devided by $f(v^*)$ for every $v^* \in V^*(G)$, then $G \in C_f 1$.*

Two above theorems will be used in the part of the proof of our results. In 2008, Zhang *et al.* [11] gave a sufficient condition for a regular graph to be in $C_f 2$. In [1], we generalized the result as stated in Theorem 3.

Theorem 3. *Let $G = (V,E)$ be a graph and let $g = \gcd(d(v):\ v \in V)$. Let $f : V \to \mathbb{N}$ be defined as $f(v) = d(v)/k$ for each $v \in V$, where k divides g. If $f(v)$ is odd for an odd number of vertices, then $G \in C_f 2$.*

Let G and H be two graphs with n and m vertices, respectively. The *corona product* of a graph G with a graph H, denoted by $G \odot H$, is a graph obtained by taking one copy of an n-vertex G and n copies of H, namely H_1, H_2, ..., H_n, and then for $i = 1,2,...,n$, joining the i-th vertex of G to every vertex of H_i. Here G is called the *center* of $G \odot H$, denoted by G^1, and H_i is called the *outer* of $G \odot H$, denoted by H_i^2.

Let C_n be a cycle on n vertices, S_n be a star on $n+1$ vertices, W_n be a wheel on $n+1$ vertices, and K_n be a complete graph on n vertices. We showed in [3] that the corona product of a cycle with either the complement of a complete graph, or a path, or another cycle is in $C_f 1$. In [2] we gave a characterization of some graphs containing wheels, namely the corona product of either the complement of a complete graph, or a path, or a star with a cycle, based on f-coloring.
As well-known, the corona product is not commutative. Hence, in this paper, we are interested in considering the corona product of C_n with S_m. In Theorem 4, we show that it is in $C_f 1$ for any f. Besides that, we consider the corona product of C_n with either W_m or K_m. By using Theorem 3, we give a characterization of the corona product of C_n with W_m based on its f-chromatic index as we state in Theorem 5. In Theorem 6, we give a necessary and sufficient condition of the corona product of C_n with K_m to be in $C_f 2$.

MAIN RESULTS

Let $G = C_n \odot S_m$ be the corona product of a cycle with a star. We label all vertices of C_n^1 by $v_1, v_2, ..., v_n$, respectively. Next, for $i = 1,2,...,n$ and $j = 1,2,...,m$, we label the center of i-th star S_m^2 by u_i and the other vertices of the i-th star S_m^2 by $w_{i,j}$. In the following theorem, we show that the corona product of a cycle with a star is in $C_f 1$ for any f.

Theorem 4. *Let $n \geq 3$, $m \geq 3$, $G = C_n \odot S_m$, and f be a function from $V(G)$ to a subset of positive integers, then $G \in C_f 1$.*

Proof.

If $\Delta_f(G) = 1$, then we can define an f-coloring of G with one color. For $\Delta_f(G) \geq 2$, we divide the proof into two cases as follows.

Case 1. $\Delta_f(G) = 2$

For $i = 1,2,...,n$ and $j = 1,2,...,m$, we color all edges $u_i v_i$ by c_1, $v_i w_{i,j}$ by c_1 and c_2 alternately, and $w_{i,j} u_i$ by c_2 and c_1 alternately such that $w_{i,j}$ is incident with two different colored edges. Next, we color all edges in $E(C_n^1)$ by c_2.

Case 2. $\Delta_f(G) \geq 3$

For $i = 1,2,...,n$ and $j = 1,2,...,m$, we color all edges $u_i w_{i,j}$ by $c_1, c_2, ..., c_{\Delta_f}$ alternately, $v_i w_{i,j}$ by $c_2, c_3, ..., c_{\Delta_f}, c_1$ alternately, and $u_i v_i$ by c_{Δ_f}. Next, we color all edges in $E(C_n^1)$ by c_1 and c_2 alternately. Finally, for odd n and for $i = 1,2,...,n$, we have to replace the color on $v_i w_{i,m}$ by c_2.

So, all edges of G have been colored by using $\Delta_f(G)$ colors and every vertex $v \in V(G)$ has at most $f(v)$ edges with a same color. □

Let $G = C_n \odot W_m$ be the corona product of a cycle with a wheel. We label all vertices of $V(C_n^1)$ by $v_1, v_2, ..., v_n$, respectively. Next, for $i = 1,2,...,n$ and $j = 1,2,...,m$, we label the center of i-th wheel W_m^2 by u_i and the other vertices of the i-th wheel W_m^2 by $w_{i,j}$. In the following theorem, we give a necessary and sufficient condition of the corona product of C_n with W_m to be in $C_f 2$.

Theorem 5. *Let $n \geq 3$, $m \geq 3$, and $G = C_n \odot W_m$.*
If n is odd and

$$f(v) = \begin{cases} \frac{m+3}{2}, & v \in V(C_n^1), \\ \frac{m+1}{2}, & v = u_i, \\ 2, & v = w_{i,j}, \end{cases}$$

then $G \in C_f 2$. Otherwise, $G \in C_f 1$.

Proof.

If the premise of the theorem is fulfilled, then $\Delta_f(G) = 2$ and $d(v) = 2f(v)$ for every $v \in V(G)$. Meanwhile, there exists odd $f(v)$ for odd number of vertices $v \in V(G)$. Hence, by Theorem 3, then $G \in C_f 2$.

If the premise of the theorem is not fulfilled, we divide the proof into two cases as follows.

Case 1. For even n

If $f(v)$ is as given above, we can construct an f-coloring by using Δ_f colors as follows. For $i = 1,2,...,n$ and $j = 1,2,...,m$, we color all edges $u_iw_{i,j}$ by $c_1, c_2, ..., c_{\Delta_f}$ alternately and $v_iw_{i,j}$ by $c_{\Delta_f}, c_1, ..., c_{\Delta_f - 1}$ alternately, and u_iv_i by c_{Δ_f}. Next, for $i = 1,2,...,n$ and $j = 1,2,...,m-1$, we color all edges $w_{i,j}w_{i,j+1}$ and $w_{i,m}w_{i,1}$ by $c_{\Delta_f - 1}$, c_{Δ_f}, c_1,... $c_{\Delta_f - 2}$ alternately. Finally, for $i = 1,2,...,n-1$, we color all edges v_iv_{i+1} and v_nv_1 by c_1 and c_2 alternately. Meanwhile, if $f(v)$ does not satisfy the above conditional, then it becomes more easy to construct an f-coloring by using Δ_f colors.

Case 2. For odd n

For $i = 1,2,...,n$ and $j = 1,2,...,m$, we color all edges $u_iw_{i,j}$ by $c_1, c_2, ..., c_{\Delta_f}$ alternately, and u_iv_i and $v_iw_{i,1}$ by c_1. For $i = 2,3,...,n$ and $j = 1,2,...,m$, we color $v_iw_{i,j}$ by $c_{\Delta_f - 1}$, $c_{\Delta_f}, c_1, ..., c_{\Delta_f - 2}$ alternately. Next, we color all edges v_iv_{i+1} and v_nv_1 by c_{Δ_f}. Finally, for $i = 1,2,...,n$ and $j = 1,2,...,m-1$, we color edge $w_{i,j}v_{i,j+1}$ and $w_{i,m}w_{i,1}$ by $c_{\Delta_f}, c_1, ..., c_{\Delta_f - 1}$ alternately.

Hence, $G \in C_f1$. □

In Theorem 6, we give a characterization of the corona product of C_n with K_m based on f-chromatic index. All vertices of C_n^1 are labeled by $v_1, v_2, ..., v_n$, and for $i = 1,2,...,n$ and $j = 1,2,...,m$, all vertices of i-th K_m^2 are labeled by $w_{i,j}$. In the following theorem, we characterize the corona product of a cycle with a complete graph based on its f-chromatic index.

Theorem 6. *Let $n \geq 3$, $m \geq 2$, and $G = C_n \odot K_m$. If n is odd, m is even, and*

$$f(v) = \begin{cases} \frac{m}{2} + 1, & v \in V(C_n^1), \\ \frac{m}{2}, & v \in V(K_m^2), \end{cases}$$

then $G \in C_f2$. Otherwise, $G \in C_f1$.

Proof.

For $i = 1,2,...,n$, let L_i be a subgraph of G which is induced by all vertices of i-th K_m^2 and v_i.

If the premise of the theorem is fulfilled, we have $\Delta_f(G) = 2$ and $d(v) = 2f(v)$ for every $v \in V(G)$. Meanwhile, there exists odd $f(v)$ for odd number of vertices $v \in V(G)$. Hence, by Theorem 3, we get $G \in C_f2$.

If the premise of the theorem is not fulfilled, we can construct an f-coloring on G by using $\Delta_f(G)$ colors. We divide the proof into two cases as follows.

Case 1. $\Delta_f(G) = m$

If $f(v) = 1$ for every $v \in V(G) \setminus V(C_n^1)$, then we divide proof into two subcases.

Subcase 1.1. For odd m

Since every L_i is a complete graph with even order, then L_i can be decomposed into m edge–disjoint matchings with $(m+1)/2$ edges each. Give a different color to each class. Next, we color all edges in $E(C_n^1)$ by using two different colors alternately. So, all edges of G have been colored by using $\Delta_f(G)$ colors and every vertex $v \in V(G)$ has at most $f(v)$ edges with a same color.

Subcase 1.2. For even m

In this case, we color all edges in $E(C_n^1)$ by using one color, namely t. Next, for $i = 1,2,...,n$ and $j = 1,2,...,m$, we color all edges which joining v_i with every vertex $w_{i,j}$ by using $m/2$ colors (except t) such that the number of colors on edges which is incident with v_i is equal to 2 for every color. Hence, every $L_i \setminus \{v_i\}$ can be decomposed into m edge-disjoint matchings. We color each *matching* by using one class of color such that the number of colors on edges which is incident with $w_{i,j}$ is equal to 1 for every color. So, we have constructed an f-coloring on G by using Δ_f colors.

If there exist $f(v) \neq 1$ for some vertices $v \in V(G) \setminus V(C_n^1)$, then we can construct an f-coloring easier than both of two subcases above.

Case 2. $\Delta_f(G) < m$

Every subgraph K_m^2 contains a complete graph K_{r+1} with $r = max\{s \leq m : s \text{ is divided by } 2\Delta_f\}$. Based on Lucas Theorem [6], a complete graph with odd order can be decomposed into some edge-disjoint Hamiltonian cycles. Hence, we decompose every K_{r+1} into $r/2$ edge-disjoint Hamiltonian cycles. Next, we partition this Hamiltonian cycles into Δ_f classes with cardinality of each is r/Δ_f. Give a different color to each class. We obtain an f-coloring on every K_{r+1} by using Δ_f colors such that every vertex is joining with r/Δ_f edges for every color.

If $m \neq 0 \pmod{r}$, then some edges in $E(K_m^2)$ are not colored yet. For $i = 1,2,...,n$, let $W_{i,j} = \{w_{i,m-r+1}, \ldots, w_{i,m-r+\Delta_f}\}$ be a subset of vertices of $K_m^2 \backslash K_{r+1}$ which are incident with edges in K_{r+1}. Next, for $i = 1,2,...,n$, we partition vertices of i-th K_{r+1} into Δ_f subsets, namely $U_{i,1}, \ldots, U_{i,\Delta_f}$. For $i = 1,2,...,n$ and $j = 1,2,...,\Delta_f$, we color the edges which is joining every vertex $w_{i,m-r+j} \in W_{i,j}$ with $U_{i,l}$ with color $(j+l) \pmod{\Delta_f}$. In case $(j+l) \equiv 0 \pmod{\Delta_f}$, we color the edges by Δ_f. It implies every vertex in $U_{i,l} \cup W_{i,j}$ has at most $(m+1)/\Delta_f$ edges with a same color. Next, for

$i = 1,2,...,n$ and $j = 1,2,...,\Delta_f$, we can color all edges which is joining vertices in $W_{i,j}$ such that the number of colors on edges which are incident with $w_{i,m-r+j}$ at most $f(w_{i,m-r+j})$.

If $m \equiv 0 \pmod r$, then $L_i \setminus \{v_i\}$ $(1 \leq i \leq n)$ is a complete graph with even order. This means that the L_i can be decomposed into $m/2$ edge-disjoint Hamiltonian cycles. Next, we recollect the Hamiltonian cycles into Δ_f classes and for every class, we color all edges by using different colors. Finally, we color all edges in $E(C_n^1)$ by using a color such that every vertex $v \in V(C_n^1)$ has at most $f(v)$ edges with a same color. □

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REFERENCES

1. Adiwijaya, A.N.M. Salman, O. Serra, D. Suprijanto, E.T. Baskoro, "Some graphs in C_f2 based on f-coloring", *submitted*.
2. Adiwijaya, A.N.M. Salman, D. Suprijanto, E.T. Baskoro, "A classification of some graphs containing wheels based on f-coloring", *East West Journal of Mathematics* Special Volume (2010) pp. 200-207.
3. Adiwijaya, A.N.M. Salman, D. Suprijanto, E.T. Baskoro, "On the f-colorings of the corona product of a cycle with some graphs", *Journal of Combinatorial Mathematics and Combinatorial Computing* **71** (2009) 235-241.
4. E. G. Coffman, M.R. Garey, D.S. Johnson, A.S. LaPaugh, "Scheduling file transfers, *SIAM Journal of Computation* **14:3** (1985), 744-780.
5. S.L. Hakimi, O. Kariv, "A generalization of edge-coloring in graphs", *Journal of Graph Theory* **10** (1986) 139-154.
6. N. Hartsfield, G. Ringel, *Pearls in Graph Theory : A Comprehensive Introduction*, Academic Press (1994), London.
7. I. Holyer, "The NP-completness of edge-coloring", *SIAM Journal of Computation* **10:4** (1981) 718 - 720.
8. J. Yu, L. Han, G. Liu, "Some result on the classification for f-colored graphs", *Proceeding ISORA (2006)* .
9. X. Zhang, G. Liu, "Some sufficient conditions for a graf to be C_f1", *Applied Mathematics Letters* **19** (2006) 38-44.
10. X. Zhang, G. Liu, "The classification of K_n on f-colorings", *Journal of Applied Mathematics and Computing* **19:1-2** (2006) 127-133.
11. X. Zhang, J. Wang, G. Liu, "The classification of regular graphs on f-colorings", *Ars Combinatoria* **86** (2008) 273-280.