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On f-colorings of the corona product of cycles with some other graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple graph and f be a function from $V(G)$ to a subset of positive integers. An *f-coloring* of G is a generalized edge-coloring such that every vertex $v \in V(G)$ has at most $f(v)$ edges colored with a same color. The minimum number of colors needed to define an f-coloring of G is called an *f-chromatic index* of G, and denoted by $\chi'_f(G)$. The f-chromatic index of G is equal to $\Delta_f(G)$ or $\Delta_f(G) + 1$, where $\Delta_f(G) = max\{\lceil d(v)/f(v) \rceil \mid v \in V(G)\}$. G is called in the *class-1*, denoted by C_f1, if $\chi'_f(G) = \Delta_f(G)$; otherwise G is called in the *class-2*, denoted by C_f2. In this paper, we showed that the corona product of a cycle with either the complement of a complete graph, or a path, or a cycle is in C_f1.

Keywords : corona product, edge coloring, f-coloring, f-chromatic index

1 Introduction and statement of results

We consider *simple graphs*, which are finite undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Let f be a function from $V(G)$ to a subset of the positive integers. An *f-coloring* of G is a coloring of edges such that each vertex v has at most $f(v)$ edges colored with a same color. The minimum number of colors needed in the f-coloring of G is called an *f-chromatic index* of G, denoted by $\chi'_f(G)$. If $f(v) = 1$ for every $v \in V(G)$, the f-coloring is the classical edge-coloring.

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A problem in the f-coloring is how to determine $\chi'_f(G)$ of a given graph G. It arises in many applications, including the network design problem, the scheduling problem, and the file transfer problem in a computer network [1, 2]. The file transfer problem in a computer network is modeled as follows. Each computer is represented by a vertex and the file transfer process every two computers is represented by an edge. Each computer v has a limit number $f(v)$ of communication ports. If we assume that the transfer time is constant for every file, we can use an f-coloring to manage transferring all files along the minimum time needed.

Let $d(v)$ be a degree of v and

$$\Delta_f(G) = \max_{v \in V(G)} \left\{ \left\lceil \frac{d(v)}{f(v)} \right\rceil \right\}. \tag{1}$$

Hakimi and Kariv [3] showed that

$$\Delta_f(G) \leq \chi'_f(G) \leq \Delta_f(G) + 1. \tag{2}$$

G is called in the *class-1*, denoted by $G \in C_f 1$, if $\chi'_f(G) = \Delta_f(G)$; otherwise G is called in the *class-2*, denoted by $G \in C_f 2$). Holyer [5] proved that the edge-coloring problem is an NP-complete. It is reduced from the 3SAT problem. Consequently, the f-coloring problem is an NP-complete problem.

Hakimi and Kariv [1] showed that any bipartite graph is in $C_f 1$. If G is a graph with even $f(v)$ for each $v \in V(G)$, then G is in $C_f 1$. Yu *et al.* [7] gave sufficient conditions for fans and wheels to be in $C_f 1$. Zhang and Liu [9] found the f-chromatic index for complete graphs and gave a classification of complete graphs on f-coloring. In 2008, Zhang *et al.* [10] presented a classification of regular graphs on f-coloring.

Let G and H be two graphs with n and m vertices, respectively. The *corona product* of a graph G with a graph H, denoted by $G \odot H$, is a graph obtained by taking one copy of an n-vertex graph G and n copies of H, namely H_1, H_2, ..., H_n, and then for $i = 1, 2, ..., n$, joining the i-th vertex of G to every vertex of H_i. Here G is called the *center* of $G \odot H$ and H is called the *outer* of $G \odot H$. In this paper, we determine the class of G which obtained by the corona product of a cycle with either the complement of a complete graph, or a path, or a cycle. Let C_m, K^c_m and P_m denote a cycle, the complement of a complete graph and a path on m vertices, respectively. Our results are shown in the following three theorems.

Theorem 1. *Let $n \geq 3$ and $m \geq 1$, $G = C_n \odot K^c_m$, and f be a function from $V(G)$ to a subset of positive integers, then $G \in C_f 1$.*

Theorem 2. *Let $n \geq 3$ and $m \geq 2$, $G = C_n \odot P_m$, and f be a function from $V(G)$ to a subset of positive integers, then $G \in C_f 1$.*

Theorem 3. *Let $n \geq 3$ and $m \geq 3$, $G = C_n \odot C_m$, and f be a function from $V(G)$ to a subset of positive integers, then $G \in C_f1$.*

Let $C = \{c_1, c_2, ..., c_{\Delta_f(G)}\}$ be a set of $\Delta_f(G)$ colors. An edge colored with $c \in C$ is called an *c-edge*. Let,

$$V_0^*(G) = \left\{v \mid \frac{d(v)}{f(v)} = \Delta_f(G), v \in V(G)\right\}, \tag{3}$$

and

$$V^*(G) = \left\{v \mid \left\lceil\frac{d(v)}{f(v)}\right\rceil = \Delta_f(G), v \in V(G)\right\}. \tag{4}$$

Zhang and Liu in [8] gave some sufficient conditions for a graph to be in C_f1 as follows.

Lemma 1. [8] (Zhang dan Liu, 2006)
Let G be a graph. If the subgraph induced by $V_0{}^*(G)$ is forest, then $G \in C_f1$.

Lemma 2. [8] (Zhang dan Liu, 2006)
Let G be a graph. If $f(v^*) \nmid d(v^*)$ for every $v^* \in V^*(G)$, then $G \in C_f1$.

The graphs $G_1 = C_5 \odot K_4^c$, $G_2 = C_5 \odot P_4$, and $G_3 = C_5 \odot C_4$ with $f(v) = 2$ for every v in the center of G and $f(v) = 1$ for every v in the outer of G do not fulfill the premise of Lemma 1 or Lemma 2. But the graphs are in C_f1 as we show in the next section.

2 Proof of the theorems

Let $V(C_n^*)$ be the vertex set of the center of G. We label all vertices in $V(C_n^*)$, respectively, by v_1, v_2,..., v_n such that $E(C_n^*) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$.

Proof of Theorem 1.
Let,

$$f^* = \min_{v \in V(C_n)} \{f(v)\}.$$

Let C^* be a multiset $\{c_i \in C|$ multiplicity of c_i is f^* for every $i\}$, where multiplicity of c_i is a number of c_i in the multiset. Now, we color all edges of G by using some colors in C^*. First, we color all edges which are incident with v_1 by using some colors in C^* such that for every $i = 1, 2, ..., \Delta_f(G)$, the number of c_i*-edges* which are incident with v_1 is at most f^*. Next, for $j = 2, 3, ..., n-1$, we can color respectively, all edges which are incident with v_j by using the colors in $C^* \setminus \{r_j\}$ where r_j is the color that has been used for $v_{j-1}v_j$, respectively.

Finally, we color all edges which incident with v_n by using colors in $C^* \setminus \{s,t\}$ where s and t are the colors that have been used for v_1v_n and $v_{n-1}v_n$, respectively. So, all edges of G have been colored by using $\Delta_f(G)$ colors and every vertex $v \in V(G)$ has at most $f(v)$ edges with a same color. □

To prove Theorem 2 and Theorem 3, for $i = 1, 2, ..., n$, we label all vertices, respectively, in P_m or C_m which adjacent to $v_i \in V(C_n^*)$ by $w_{i,j}$ for $j = 1, 2, ..., m$. Next, we proof that the corona product of a cycle with a path is in C_f1

Proof of Theorem 2.
It is trivial for case $\Delta_f(G) = 1$. For $\Delta_f(G) \geq 2$, we divide the proof into three cases as follows.

Case 1. $\Delta_f(G) = 2$
In this case, we construct an f-coloring of G as follows. For $i = 1, 2, ..., n$, $j = 1, 2, ..., \lceil \frac{m+2}{2} \rceil$, and $\lceil \frac{m+2}{2} \rceil \leq k \leq m-1$, we color all edges $v_iw_{i,j}$ and $w_{i,k}w_{i,k+1}$ by c_1. Finally, we color the other edges by c_2.

Case 2. $\Delta_f(G) = 3$
First, for $i = 1, 2, ..., \lfloor \frac{n}{2} \rfloor$, we color all edges $v_{2i-1}v_{2i}$ by c_1 and for $i = 1, 2, ..., \lceil \frac{n-2}{2} \rceil$, we color all edges $v_{2i}v_{2i+1}$ by c_2. Next, for $i = 2, 3, ..., n-1$ and $j = 1, ..., m$, we color all edges $v_iw_{i,j}$ by c_3, c_1, c_2, alternately. For $i = 2, 3, ..., n-1$ and $j = 1, 2, ..., m-1$, we color all edges $w_{i,j}w_{i,j+1}$ by c_2, c_3, c_1, alternately.
Next, we divide the proof into two subcases.

Subcase 2.1. If n is even, for $i = 1$ or $i = n$ and $j = 1, 2, ..., m$, we color all edges $v_iw_{i,j}$ by c_3, c_1, c_2, alternately. Next, for $j = 1, 2, ..., m-1$, we color $w_{1,j}w_{1,j+1}$ or $w_{n,j}w_{n,j+1}$ by c_2, c_3, c_1, alternately. Finally, we color the edge v_nv_1 by c_2.

Subcase 2.2. If n is odd, for $j = 1, 2, ..., m$, we color edges $v_1w_{1,j}$ by c_2, c_3, c_1, alternately, and $v_nw_{n,j}$ by c_1, c_2, c_3, alternately. Next, for $j = 1, 2, ..., m-1$, we color $w_{1,j}w_{1,j+1}$ by c_1, c_2, c_3, alternately and $w_{n,j}w_{n,j+1}$ by c_3, c_1, c_2, alternately. Finally, we color the edge v_nv_1 by c_3.

Case 3. $\Delta_f(G) \geq 4$
If $v \in V^*$, then $v \in V(C_n^*)$. If $V_0^* = \varnothing$, by Lemma 2, then $G \in C_f1$. If $V_0^* \subset V(C_n^*)$, by Lemma 1, then $G \in C_f1$. If $V_0^* = V(C_n^*)$, we color all

edges as follows. For $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$, we color $v_i w_{i,j}$ by c_1, c_2,..., $c_{\Delta_f(G)}$, alternately. If n is odd, then we replace the color of the first of $c_{\Delta_f(G)}$-edges which incident with v_n by $c_{\Delta_f(G)-1}$. For $i = 1, 2, ..., n$ and $j = 1, ..., m-1$, we color all edges $w_{i,j}w_{i,j+1}$ by c_3, c_4,..., $c_{\Delta_f(G)}$, c_1, c_2, alternately. Finally, for $i = 1, 2, ..., n-1$, we color all edges $v_i v_{i+1}$ by $c_{\Delta_f(G)-1}$ and $c_{\Delta_f(G)}$, alternately, and $v_n v_1$ by $c_{\Delta_f(G)}$.
So, all edges of G have been colored by using $\Delta_f(G)$ colors and every vertex $v \in V(G)$ has at most $f(v)$ edges with a same color. □

In the next, we prove that the corona product of a cycle with a cycle is in $C_f 1$.

Proof of Theorem 3.
For case $\Delta_f(G) = 1$, it is trivial since we can define an f-coloring of G with one color. For $\Delta_f(G) \geq 2$, we divide the proof into four cases as follows.

Case 1. $\Delta_f(G) = 2$
In this case, we construct an f-coloring of G as follows. First, we color all edges in $E(C_n^*)$ by c_1. For $i = 1, 2, ..., n$ and $j = 1, 2, ..., m-1$, we color $v_i w_{i,j}$ by c_2, c_1, alternately, and $v_i w_{i,m}$ by c_2. Finally, for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m-1$, we color $w_{i,j}w_{i,j+1}$ by c_1, c_2, alternately, and $w_{i,m}w_{i,1}$ by c_1.

Case 2. $\Delta_f(G) = 3$
In this case, the proof is divided into three subcases as follows.

Subcase 2.1 $m = 0 \pmod 3$
First, for $i = 1, 2, ..., \lfloor \frac{n}{2} \rfloor$, we all edges $v_{2i-1}v_{2i}$ by color c_1 and for $i = 1, 2, ..., \lceil \frac{n-2}{2} \rceil$, we color all edges $v_{2i}v_{2i+1}$ by c_2. For even n, we color the last edge $v_n v_1$ by c_2. For odd n, we color the last edge $v_n v_1$ by c_3. For $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$, we color $v_i w_{i,j}$ by c_1, c_2, c_3, alternately. Finally, for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m-1$, we color all edges $w_{i,j}w_{i,j+1}$ by c_3, c_1, c_2, alternately.

Subcase 2.2 $m = 1 \pmod 3$
First, for $i = 1, 2, ..., n$ and $j = 3, ..., m-2$, we color $v_i w_{i,j}$ by c_1, c_2, c_3, alternately. We color $v_i w_{i,1}$, $v_i w_{i,2}$, $v_i w_{i,m-1}$, $v_i w_{i,m}$ for $i = 1, 2, ..., n$ by c_3, c_3, c_1, and c_1, respectively. For $i = 1, 2, ..., n$ and $k = 1, 2, ..., m-1$, we color all edges $w_{i,k}w_{i,k+1}$ by c_1, c_2, c_3, alternately, and $w_{i,m}w_{i,1}$ by c_2. Finally, we color all edges in $E(C_n^*)$ by c_2.

Subcase 2.3 $m = 2 \pmod 3$
For $i = 1, 2, ..., n$ and $j = 1, 2, ..., 5$, we color $v_i w_{i,j}$ by c_1, c_2, c_2, c_2, and c_3, respectively. For $i = 1, 2, ..., n$ and $6 \leq j \leq m$, we color edges $v_i w_{i,j}$ by c_1, c_2, c_3, alternately. Next, for $k = 1, 2, 3$, we color $w_{i,k} w_{i,k+1}$ by c_3, c_1, c_3, respectively. For $4 \leq k \leq m$, we color $w_{i,k} w_{i,k+1}$ by c_1, c_2, c_3, alternately, and $w_m w_1$ by c_2. Finally, we color all edges in $E(C_n^*)$ by c_1, c_3, alternately.

Case 3. $\Delta_f(G) = 4$
If $v \in V^*$, then $v \in V(C_n^*)$. If $V_0^* = \varnothing$, by Lemma 2, then $G \in C_f 1$. If $V_0^* \subset V(C_n^*)$, by Lemma 1, then $G \in C_f 1$. If $V_0^* = V(C_n^*)$, we color all edges as follows. For $i = 1, 2, ..., n$ and $j = 2, 3, ..., m-1$, we color $v_i w_{i,j}$ by c_1, c_2, c_3, c_4, alternately, $v_i w_{i,1}$ by c_1, and $v_i w_{i,m}$ by c_2. If n is odd, then we replace the color of the first of c_4-edges which incident with v_n by c_3. For $i = 1, 2, ..., n$ and $j = 1, 2, ..., m-2$, we color all edges $w_{i,j} w_{i,j+1}$ by c_2, c_3, c_4, c_1, alternately. Next, for $i = 1, ..., n$, we color $w_{i,m-1} w_{i,m}$ by c_3 and $w_{i,m} w_{i,1}$ by c_4. Finally, for $i = 1, 2, ..., n-1$, we color all edges $v_i v_{i+1}$ by c_3, c_4, alternately, and $v_n v_1$ by c_4.

Case 4. $\Delta_f(G) \geq 5$
By the same argument in the Case 3, if $V_0^* = \varnothing$ or $V_0^* \subset V(C_n^*)$, we obtain $G \in C_f 1$. If $V_0^* = V(C_n^*)$, we color all edges as follows. For $i = 1, 2, ..., n$ and $j = 1, ..., m$, we color $v_i w_{i,j}$ by $c_1, c_2, ..., c_{\Delta_f(G)}$, alternately. If n is odd, then we replace the color of the first of $c_{\Delta_f(G)}$-edges which incident with v_n by $c_{\Delta_f(G)-1}$. For $i = 1, 2, ..., n$ and $j = 1, 2, ..., m-1$, we color all edges $w_{i,j} w_{i,j+1}$ by c_3, c_4, ..., $c_{\Delta_f(G)}$, c_1, c_2, alternately, and the edge $w_{i,m} w_{i,1}$ by $c_{\Delta_f(G)}$. Finally, for $i = 1, 2, ..., n-1$, we color all edges $v_i v_{i+1}$ by $c_{\Delta_f(G)-1}$, $c_{\Delta_f(G)}$, alternately, and $v_n v_1$ by $c_{\Delta_f(G)}$.
So, all edges of G have been colored by using $\Delta_f(G)$ colors and every vertex $v \in V(G)$ has at most $f(v)$ edges with a same color. $\square$

Remark : Base on the proof of Theorem 3 Subcase 2.3, we conclude that $W_m = C_m + \{w\}$ is in $C_f 1$, if $m = 2 \pmod 3$ and $f(v) \neq 1$ for some $v \in V(C_m)$. So, the conjecture of Yu et al. in [7], namely: a wheel W_m is in $C_f 2$ if $\Delta_f(W_m) = 3$ and $d(w) = 2 \pmod 3$ with w is core of W_m, is not correct.

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