The Ramsey Number for a Linear Forest versus Two Identical Copies of Complete Graphs

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Abstract. Let H be a graph with the chromatic number h and the chromatic surplus s. A connected graph G of order n is called H*-good* if $R(G,H) = (n-1)(h-1)+s$. We show that P_n is $2K_m$-good for $n \geq 3$. Furthermore, we obtain the Ramsey number $R(L, 2K_m)$, where L is a linear forest. In addition, we also give the Ramsey number $R(L, H_m)$ which is an extension for $R(kP_n, H_m)$ proposed by Ali et al. [1], where H_m is a cocktail party graph on $2m$ vertices.

Keywords: (G,H)-free, H-good, linear forest, Ramsey number, path.

1 Introduction

Throughout this paper we consider finite undirected simple graphs. For graphs G, H where H is a subgraph of G, we define $G - H$ as the graph obtained from G by deleting the vertices of H and all edges incident to them. The order of a graph G, $|G|$, is the number of vertices of G. The minimum (maximum) degree of G is denoted by $\delta(G)$ ($\Delta(G)$). Let A be a subset of vertices of a graph G, a graph $G[A]$ represents *the subgraph induced* by A in G. We denote a tree on n vertices by T_n, a path on n vertices by P_n and a complete graph on n vertices by K_n. *A cocktail party graph*, H_n, is a graph obtained by removing n independent edges from a complete graph of order $2n$. Two identical copies of complete graphs is denoted by $2K_n$.

Given graphs G and H, a graph F is called (G,H)*-free* if F contains no subgraph isomorphic to G and the complement of F, $\overline{F}$, contains no subgraph isomorphic to H. Any (G,H)-free graph on n vertices is denoted by (G,H,n)-free. *The Ramsey number* $R(G,H)$ is defined as the smallest natural number n such that no (G,H,n)-free graph exists.

Ramsey Theory studies the conditions when a combinatorial object contains necessarily some smaller given object. The role of Ramsey number is to quantify

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some of the general existential theorems in Ramsey theory. The Ramsey number $R(G, H)$ is called *the classical Ramsey number* if both G and H are complete graphs and in short denoted by $R(p, q)$ when $G \simeq K_p$ and $H \simeq K_q$. It is a challenging problem to find the exact values of $R(p, q)$. Until now, according to the survey in Radziszowski [10] there are only nine exact values of $R(p, q)$ which have been known, namely for $p = 3$, $q = 3, 4, 5, 6, 7, 8, 9$ and $p = 4$, $q = 4, 5$. In the relation with the theory of complexity, Burr [6] stated that for given graphs G, H and positive integer n, determining whether $R(G, H) \leq n$ holds is NP-hard. Furthermore in [11], we can find a rare natural example of a problem higher than NP-hard in the polynomial hierarchy of computational complexity theory, that is, Ramsey arrowing is $\prod_2^p$-complete.

Since it is very difficult to determine $R(p, q)$, one turns out to consider the problem of Ramsey numbers concerning the general graphs G and H, which are not necessarily complete, such as the Ramsey numbers for path versus path, tree versus complete graph, path versus cocktail party graph and so on. This makes the problem on the graphs Ramsey number become more interesting, especially for the union of graphs.

Let k be a positive integer and G_i be a connected graph with the vertex set V_i and the edge set E_i for $i = 1, 2, ..., k$. *The union of graphs*, $G \simeq \bigcup_{i=1}^k G_i$, has the vertex set $V = \bigcup_{i=1}^k V_i$ and the edge set $E = \bigcup_{i=1}^k E_i$. If $G_1 \simeq G_2 \simeq ... \simeq G_k \simeq F$, where F is an arbitrary connected graph, then we denote the union of graphs by kF. The union of graphs is called *a forest* if G_i is isomorphic to T_{n_i} for every i. In particular, if $G_i \simeq P_{n_i}$ for every i then the union of graphs is called *a linear forest*, — denoted by L.

Let H be a graph with the chromatic number h and the chromatic surplus s. The chromatic surplus of H, s, is the minimum cardinality of a color class taken over all proper h colorings of H. A connected graph G of order n is called *H-good* if $R(G, H) = (n - 1)(h - 1) + s$. In particular, for tree T_n versus complete graph K_m, Chvátal [5] showed that T_n is K_m-good with $s = 1$. Other results concerning H-good graphs with the chromatic surplus one can be found in Radziszowski [10]. Recent results on the Ramsey number for the union of graphs consisting of H-good components with $s = 1$ can be found in [2], [3], [8]. Other results concerning the Ramsey number for the union of graphs containing no H-good components can be seen in [9], [12], [13].

However, there are many graphs that have the chromatic surplus greater than one. For example, the graphs $2K_m$ and H_m have the same chromatic surplus, that is 2. Therefore, in this paper we show that P_n is $2K_m$-good for $n \geq 3$. Based on this result, we obtain the Ramsey number $R(L, 2K_m)$. In addition, we give the Ramsey number $R(L, H_m)$ which is an extension for $R(kP_n, H_m)$ proposed by Ali et al. [1].

Let us note firstly the previous theorems and lemma used in the proof of our results.

Theorem 1. *(Gerencser and Gyarfas [7]).* $R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$, *for* $n \geq m \geq 2$.

Theorem 2. *(Ali et al. [1]). Let P_n be a path on n vertices and H_m be a cocktail party graph on $2m$ vertices. Then, $R(kP_n, H_m) = (n-1)(m-1) + (k-1)n + 2$, for $n, m \geq 3$ and $k \geq 1$.*

Lemma 1. *(Bondy [4]). Let G be a graphs of order n. If $\delta(G) \geq \frac{n}{2}$ then either G is pancyclic or n is even and $G \simeq K_{\frac{n}{2},\frac{n}{2}}$.*

2 The Ramsey Goodness for P_n versus $2K_m$

The theorem in this section deals with the Ramsey goodness for a path versus two identical copies of complete graphs. First we need to prove the following two lemmas.

Lemma 2. *Let t, n be positive integers and P_n be a path on $n \geq 2$ vertices and K_2 be a complete graph on 2 vertices. Then,*

$$R(P_n, tK_2) = \begin{cases} n+t-1 & \text{if } t \leq \lfloor \frac{n}{2} \rfloor, \\ 2t + \lfloor \frac{n}{2} \rfloor - 1 & \text{if } t > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Proof. We separate the proof into two cases.

Case 1. $t \leq \lfloor \frac{n}{2} \rfloor$

To prove the upper bound $R(P_n, tK_2) \leq n+t-1$ we use induction on t. For $t = 1$, the assertion is hold from the trivial Ramsey number $R(P_n, K_2) = n$. Assume that the lemma is true for $t-1$. We shall show that the lemma is also valid for t. Let F be an arbitrary graph on $n+t-1$ vertices containing no P_n. We will show that $\overline{F}$ contains tK_2. By induction on t, $\overline{F}$ contains $(t-1)K_2$. Let $B = \{a_1, b_1, ..., a_{t-1}, b_{t-1}\}$ be the vertex set of $(t-1)K_2$ in $\overline{F}$, where a_ib_i are the independent edges in $\overline{F}$ for $i = 1, 2, ..., t-1$. By a contrary, suppose that $\overline{F}$ contains no tK_2. Let $A = V(F)\backslash B$, clearly $|A| = n-t+1$. Then the subgraph $F[A]$ of F forms a K_{n-t+1}. Otherwise, if there exists two independent vertices in $F[A]$, say x and y, then the vertex set $\{x, y\} \cup B$ forms a tK_2 in $\overline{F}$.

Let us now consider the relation of the vertices in $F[A]$ and B. If a_i (or b_i) is not adjacent to one vertex in $F[A]$ then b_i (or a_i) must be adjacent to all other vertices in $F[A]$ since otherwise we will get two independent edges between $\{a_i, b_i\}$ and $F[A]$ in $\overline{F}$, together with B, the vertices form a tK_2 in $\overline{F}$. Without loss of generality, we may assume that each b_i is adjacent to all but at most one vertex in $F[A]$. Let us consider the subgraph $F[D]$ of F with $D = A \cup \{b_1, b_2, ..., b_{t-1}\}$. Clearly, the subgraph $F[D]$ has n vertices and $\delta(F[D]) \geq n-t$. Since $t \leq \lfloor \frac{n}{2} \rfloor$ then $\delta(F[D]) \geq \lceil \frac{n}{2} \rceil \geq \frac{n}{2}$. Lemma 1 now applies, the subgraph $F[D]$ contains a cycle of order n. This is a contradiction because there is no P_n in F. Therefore, $\overline{F}$ contains tK_2.

Next it can be verified that $K_{n-1} \cup K_{t-1}$ is a (P_n, tK_2)-free graph on $n+t-2$ vertices and hence $R(P_n, tK_2) \geq n+t-1$.

Case 2. $t > \lfloor \frac{n}{2} \rfloor$

In this case, Theorem 1 implies that $R(P_n, tK_2) \leq R(P_n, P_{2t}) = 2t + \lfloor \frac{n}{2} \rfloor - 1$.

Conversely, $K_{\lfloor \frac{n}{2} \rfloor -1} + \overline{K}_{2t-1}$ is a (P_n, tK_2)-free graph on $2t + \lfloor \frac{n}{2} \rfloor - 2$ vertices and therefore $R(P_n, tK_2) \geq 2t + \lfloor \frac{n}{2} \rfloor - 1$. The proof is now complete. □

Lemma 3. *Let K_m be a complete graph on m vertices and P_3 be a path on 3 vertices. Then, $R(P_3, 2K_m) = 2m$.*

Proof. We prove the upper bound $R(P_3, 2K_m) \leq 2m$ by induction on m. We have $R(P_3, 2K_2) = 4$ from Lemma 2 and therefore the assertion holds for $m = 2$. Now assume that the assertion is true for $m-1$, namely $R(P_3, 2K_{m-1}) \leq 2(m-1)$. We shall show that the lemma is also valid for m. Let F be an arbitrary graph on $2m$ vertices containing no P_3. We will show that $\overline{F}$ contains $2K_m$. By trivial Ramsey number $R(P_2, 2K_m) = 2m$, F contains P_2 or $\overline{F}$ contains $2K_m$. If $\overline{F}$ contains $2K_m$ then the proof is complete. Now consider that F contains P_2 and let u and v be the two vertices of P_2. Clearly, the subgraph $F - P_2$ of F has $2(m-1)$ vertices. By induction hypothesis on m, the complement of $F - P_2$ contains $2K_{m-1}$. Since F contains no P_3 then the vertices u and v are not adjacent to any vertices in $2K_{m-1}$ and hence of course $\{u, v\}$ and $2K_{m-1}$ form a $2K_m$ in $\overline{F}$.

It is easy to verify that $\overline{K}_{2m-1}$ is $(P_3, 2K_m)$-free graph on $2m-1$ vertices and hence of course $R(P_3, 2K_m) \geq 2m$. This concludes the proof. □

Now we are ready to prove the following theorem.

Theorem 3. *Let P_n be a path of order $n \geq 3$ and K_m be a complete graph of order $m \geq 2$. Then, $R(P_n, 2K_m) = (n-1)(m-1) + 2$.*

Proof. Let us consider a graph $G \simeq (m-1)K_{n-1} \cup K_1$. It can be verified that G contains no P_n and $\overline{G}$ contains no $2K_m$. Therefore, G is $(P_n, 2K_m)$-free graph on $(n-1)(m-1)+1$ vertices and of course $R(P_n, 2K_m) \geq (n-1)(m-1)+2$.

Next we prove the upper bound $R(P_n, 2K_m) \leq (n-1)(m-1)+2$ by induction on $n+m$. For $m = 2$ and $n = 3$, the assertion holds from Lemmas 2 and 3, respectively. Now assume that the assertion is true for $n+m-1$, namely

(1). $R(P_{n-1}, 2K_m) \leq (n-2)(m-1) + 2$ and
(2). $R(P_n, 2K_{m-1}) \leq (n-1)(m-2) + 2$.

We shall show that the theorem is also valid for $n+m$. Let F be an arbitrary graph on $(n-1)(m-1)+2$ vertices. We will show that F contains P_n or $\overline{F}$ contains $2K_m$. By induction hypothesis on n in (1), F contains P_{n-1} or $\overline{F}$ contains $2K_m$. If $\overline{F}$ contains $2K_m$ then it finishes the proof. Now consider that F contains P_{n-1} and let u and v be the two end vertices of the path P_{n-1}. It can be verified that the subgraph $F - P_{n-1}$ of F has $(n-1)(m-2)+2$ vertices. By induction hypothesis on m in (2), the subgraph $F - P_{n-1}$ contains P_n or the complement of $F - P_{n-1}$ contains $2K_{m-1}$. If the subgraph $F - P_{n-1}$ contains P_n then the proof is done. Therefore, the complement of $F - P_{n-1}$ contains $2K_{m-1}$. If u or v is adjacent to one vertex in $2K_{m-1}$ then we have P_n in F. Conversely, if u and v are not adjacent to any vertices in $2K_{m-1}$ then we obtain $2K_m$ in $\overline{F}$. So, $R(P_n, 2K_m) = (n-1)(m-1) + 2$. The proof is done. □

3 The Ramsey Numbers for L versus $2K_m$ or H_m

The following two theorems deal with the Ramsey numbers for the union of graphs containing H-good components with $s = 2$. In particular, for a linear forest versus $2K_m$ or H_m. First we need to prove the following lemma.

Lemma 4. *Let K_m be a complete graph on $m \geq 2$ vertices and P_n be a path on $n \geq 3$ vertices. Then, $R(kP_n, 2K_m) = (n-1)(m-1) + (k-1)n + 2$, for $k \geq 1$.*

Proof. We prove the upper bound $R(kP_n, 2K_m) \leq (n-1)(m-1)+(k-1)n+2$ by induction on k. For $k = 1$, the assertion holds from Theorem 3. Assume that the assertion is true for $k-1$, that is $R((k-1)P_n, 2K_m) \leq (n-1)(m-1)+(k-2)n+2$. We shall show that the lemma is also valid for k. Let F be an arbitrary graph on $(n-1)(m-1)+(k-1)n+2$ vertices and suppose that $\overline{F}$ contains no $2K_m$. We will show that F contains kP_n. By inductive hypothesis, F contains $(k-1)P_n$. Thus, the subgraph $F-(k-1)P_n$ of F has $(n-1)(m-1)+2$ vertices. Now by Theorem 3, the subgraph $F-(k-1)P_n$ contains P_n and hence we obtain kP_n in F.

Next construct a graph $G \simeq (m-2)K_{n-1} \cup K_{kn-1} \cup K_1$. It is not hard to verify that G is a $(kP_n, 2K_m)$-free graph on $(n-1)(m-1)+(k-1)n+1$ vertices and therefore $R(kP_n, 2K_m) \geq (n-1)(m-1)+(k-1)n+2$. The proof is now complete. □

Now we are ready to prove the following theorem.

Theorem 4. *For integers $k \geq 1$, let $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 3$ be integers. Let K_m be a complete graph on $m \geq 2$ vertices, P_{n_i} be a path on n_i vertices for $i = 1, 2, ..., k$ and $L \simeq \bigcup_{i=1}^{k} l_i P_{n_i}$. Then,*

$$R(L, 2K_m) = \max_{1\leq i\leq k} \left\{(n_i-1)(m-2) + \sum_{j=i}^{k} l_j n_j + 1\right\}, \tag{1}$$

where l_i is the number of paths of order n_i in L.

Proof. Let $t = (n_{i_0}-1)(m-2) + t_0 + 1$ be the maximum of the right side of the Eq. (1) achieved for i_0, where $t_0 = \sum_{j=i_0}^{k} l_j n_j$. Now construct a graph $F \simeq (m-2)K_{n_{i_0}-1} \cup K_{t_0-1} \cup K_1$. Since $n_i \geq 3$ for every $i = 1, 2, ..., k$ then F does not contain at least one component P_{n_i} of L and hence F contains no L. Note that $\overline{F} \simeq K_{n_{i_0}-1,...,n_{i_0}-1,t_0-1,1}$ is a complete m-partite graph, where the smallest partite consists of one vertex. So, there is no $2K_m$ in $\overline{F}$. Thus, F is a $(L, 2K_m)$-free graph on $t-1$ vertices and therefore $R(L, 2K_m) \geq t$.

In order to show that $R(L, 2K_m) \leq t$ we argue as follows. Let U be an arbitrary graph on t vertices and suppose that $\overline{U}$ contains no $2K_m$. We will show that U contains L by induction on k. For $k = 1$, the theorem is true from Lemma 4. Let us assume that the theorem holds for $k-1$. We shall show that the theorem is also valid for k. Note that $t \geq (n_k - 1)(m-1) + (l_k - 1)n_k + 2$. Thus, by Lemma 4, U contains $l_k P_{n_k}$. By definition of t, we obtain the following fact.

$$|U - l_k P_{n_k}| = t - l_k n_k \geq \max_{1 \leq i \leq k-1} \left\{ (n_i - 1)(m-2) + \sum_{j=i}^{k-1} l_j n_j + 1 \right\}. \quad (2)$$

By induction hypothesis, the subgraph $U - l_k P_{n_k}$ of U contains $\bigcup_{i=1}^{k-1} l_i P_{n_i}$ and together with $l_k P_{n_k}$ we have $L \simeq \bigcup_{i=1}^{k} l_i P_{n_i}$ in U. Therefore, $R(L, 2K_m) = t$. This completes the proof. □

The following theorem is an extension of Theorem 2 proposed by Ali et al. in [1].

Theorem 5. *Let $k \geq 1$ and $m \geq 3$. Let $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 3$ be integers. Let H_m be a cocktail party graph on $2m$ vertices, P_{n_i} be a path on n_i vertices for $i = 1, 2, ..., k$ and $L \simeq \bigcup_{i=1}^{k} l_i P_{n_i}$. Then,*

$$R(L, H_m) = \max_{1 \leq i \leq k} \left\{ (n_i - 1)(m-2) + \sum_{j=i}^{k} l_j n_j + 1 \right\}, \quad (3)$$

where l_i is the number of paths of order n_i in L.

Proof. Let $t = (n_{i_0} - 1)(m-2) + t_0 + 1$ be the maximum of the right side of the Eq. (3) achieved for i_0 with $t_0 = \sum_{j=i_0}^{k} n_j$. Now we construct a graph $G \simeq (m-2)K_{n_{i_0}-1} \cup K_{t_0-1} \cup K_1$. Since $n_i \geq 3$ for every $i = 1, 2, ..., k$ then G does not contain at least one component P_{n_i} of L and hence G contains no L. Note that $\overline{G} \simeq K_{n_{i_0}-1,...,n_{i_0}-1,t_0-1,1}$ is a complete m-partite graph which the smallest partite has one vertex. Then, $\overline{G}$ contains no H_m. So, G is a (L, H_m)-free graph on $t-1$ vertices and hence $R(L, H_m) \geq t$.

We will prove the upper bound $R(L, H_m) \leq t$ by induction on k. Let F be an arbitrary graph of order t and suppose that $\overline{F}$ contains no H_m. We shall show that F contains L. From Theorem 2, we can see that the assertion is true for $k = 1$. Now let us assume that the theorem also holds for $k > 1$. Note that $t \geq (n_k - 1)(m-1) + 2$. So, by Theorem 2, F contains $l_k P_{n_k}$. From the definition of t, we get the following fact.

$$|F - l_k P_{n_k}| = t - l_k n_k \geq \max_{1 \leq i \leq k-1} \left\{ (n_i - 1)(m-2) + \sum_{j=i}^{k-1} l_k n_j + 1 \right\}. \quad (4)$$

By induction hypothesis, the subgraph $F - l_k P_{n_k}$ of F contains $\bigcup_{i=1}^{k-1} l_i P_{n_i}$ and hence $F \supseteq \bigcup_{i=1}^{k} l_i P_{n_i}$. This completes the proof. □

4 Conclusion

To conclude this paper let us present the following two conjectures to work on.

Conjecture 1. *Let T_n be a tree on $n \geq 3$ vertices and K_m be a complete graph on $m \geq 2$ vertices. Then, $R(T_n, 2K_m) = (n-1)(m-1)+2$.*

Conjecture 2. *For integers $k \geq 1$, let $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 3$ be integers. Let K_m be a complete graph on $m \geq 2$ vertices, T_{n_i} be a tree on n_i vertices for $i = 1, 2, ..., k$ and $F \simeq \bigcup_{i=1}^{k} l_i T_{n_i}$. Then,*

$$R(F, 2K_m) = \max_{1 \leq i \leq k} \left\{ (n_i - 1)(m-2) + \sum_{j=i}^{k} n_j + 1 \right\}, \tag{5}$$

where l_i is the number of trees of order n_i in F.

References

1. Ali, K., Baig, A.Q., Baskoro, E.T.: On the Ramsey number for a linear forest versus a cocktail party graph. J. Combin. Math. Combin. Comput. 71, 173–177 (2009)
2. Baskoro, E.T., Hasmawati, Assiyatun, H.: The Ramsey number for disjoint unions of trees. Discrete Math. 306, 3297–3301 (2006)
3. Bielak, H.: Ramsey numbers for a disjoint of some graphs. Appl. Math. Lett. 22, 475–477 (2009)
4. Bondy, J.A.: Pancyclic graph. J. Combin. Theory Ser. B 11, 80–84 (1971)
5. Chvátal, V.: Tree complete graphs Ramsey number. J. Graph Theory 1, 93 (1977)
6. Burr, S.A.: Determining generalized Ramsey numbers is NP-hard. Ars Combin. 17, 21–25 (1984)
7. Gerencser, L., Gyarfas, A.: On Ramsey-Type problems, Annales Universitatis Scientiarum Budapestinensis. Eotvos Sect. Math. 10, 167–170 (1967)
8. Hasmawati, Baskoro, E.T., Assiyatun, H.: The Ramsey number for disjoint unions of graphs. Discrete Math. 308, 2046–2049 (2008)
9. Hasmawati, Assiyatun, H., Baskoro, E.T., Salman, A.N.M.: Ramsey numbers on a union of identical stars versus a small cycle. In: Ito, H., Kano, M., Katoh, N., Uno, Y. (eds.) KyotoCGGT 2007. LNCS, vol. 4535, pp. 85–89. Springer, Heidelberg (2008)
10. Radziszowski, S.P.: Small Ramsey numbers. Electronic J. Combin. DS1 12 (August 4, 2009), http://www.combinatorics.org
11. Schaefer, M.: Graph Ramsey theory and the polynomial hierarchy. J. Comput. Sys. Sci. 62, 290–322 (2001)
12. Sudarsana, I.W., Baskoro, E.T., Assiyatun, H., Uttunggadewa, S.: On the Ramsey numbers of certain forest respect to small wheels. J. Combin. Math. Combin. Comput. 71, 257–264 (2009)
13. Sudarsana, I.W., Baskoro, E.T., Assiyatun, H., Uttunggadewa, S., Yulianti, L.: The Ramsey number for the union of graphs containing at least one H-good component (Under revision for publication in Utilitas Math.)