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A CLASSIFICATION OF SOME GRAPHS CONTAINING WHEELS BASED ON f-COLORINGS

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Abstract

Let $G = (V(G), E(G))$ be a simple graph and f be a function from V to a subset of positive integers. An *f-coloring* of G is a generalized edge-coloring such that every vertex $v \in V$ has at most $f(v)$ edges colored with a same color. The minimum number of colors needed to define an f-coloring of G is called an *f-chromatic index* of G, denoted by $\chi'_f(G)$. A problem in the f-coloring is how to determine $\chi'_f(G)$ of a given graph G. Based on the f-chromatic index, a graph G can be either in the C_f1 or C_f2. In this paper, we consider a graph containing wheels, especially the corona product of either the complement of a complete graph, or a path, or a star with a cycle. We give a classification of these graphs based on f-colorings.

1. Introduction

In this paper, we deal with *simple graphs* which are finite undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a graph with the

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vertex set $V(G)$ and the edge set $E(G)$. Let f be a function from $V(G)$ to a subset of positive integer. An *f-coloring* of G is a coloring of edges such that each vertex v has at most $f(v)$ edges colored with a same color. The minimum number of colors needed to define an f-coloring of G is called an *f-chromatic index* of G, denoted by $\chi'_f(G)$.

A problem in the f-coloring is how to determine $\chi'_f(G)$ of a given graph G. It arises in many applications, including network design problem, the scheduling problems, and file transfer problems in a computer network [4]. The file transfer problem in a computer network is modeled as follows. Each computer is represented by a vertex and every file transfer process between two computers is represented by an edge. Each computer v has a limit number $f(v)$ of communication ports. If we assume that the transfer time is constant for every file, we can use an f-coloring to manage transferring all files along the minimum time needed.

Let,

$$\Delta_f(G) = \max_{v\in V(G)} \left\{ \left\lceil \frac{d(v)}{f(v)} \right\rceil \right\}, \tag{1}$$

where $d(v)$ is the degree of v.
Hakimi and Kariv [5] showed that

$$\Delta_f(G) \le \chi'_f(G) \le \Delta_f(G) + 1. \tag{2}$$

G is called in the *class-1*, denoted by $G \in C_f1$, if $\chi'_f(G) = \Delta_f(G)$; otherwise G is called in the *class-2*, denoted by $G \in C_f2$. Holyer [6] proved that the edge-coloring problem is an NP-complete. It is reduced from the 3SAT problem. Consequently, the f-coloring problem is an NP-complete problem.

Hakimi and Kariv [5] showed that any bipartite graph is in C_f1. In [2], we showed that any helm graphs, any gear graphs and some friendship graphs are in C_f1. If G is a graph with even $f(v)$ for each $v \in V(G)$, then G is in C_f1. Yu et al. [8] gave sufficient conditions for fans and wheels to be in C_f1. Zhang and Liu [10] found the f-chromatic index for complete graphs and gave a classification of complete graphs based on f-colorings. In 2008, Zhang *et al.* [11] presented a classification of regular graphs based on f-coloring.

Let,

$$V_0^*(G) = \left\{ v \mid \frac{d(v)}{f(v)} = \Delta_f(G), v \in V(G) \right\}, \tag{3}$$

and

$$V^*(G) = \left\{ v \mid \left\lceil \frac{d(v)}{f(v)} \right\rceil = \Delta_f(G), v \in V(G) \right\}. \tag{4}$$

Zhang and Liu in [9] gave some sufficient conditions for a graph to be in C_f1 as follows.

Theorem 1.1 [9] *Let G be a graph. If the subgraph induced by $V_0{}^*(G)$ is forest, then $G \in C_f 1$.*

Theorem 1.2 [9] *Let G be a graph. If $d(v^*)$ is not devided by $f(v^*)$ for every $v^* \in V^*(G)$, then $G \in C_f 1$.*

We use $[a, b]$ instead of $\{x \in \mathbb{N} | a \leq x \leq b\}$. Let G and H be two graphs with n and m vertices, respectively. The *corona product* of G with H, denoted by $G \odot H$, is a graph obtained by taking one copy of G and n copies of H, namely H_1, H_2, ..., H_n, and then for $i \in [1, n]$, joining the i-th vertex of G to every vertex of H_i. Here G is called the *center* of $G \odot H$ and H is called the *outer* of $G \odot H$. In this paper, a vertex set and an edge set in the center of $G \odot H$ is denoted by $V(G)$ and $E(G)$, respectively. A vertex set and an edge set in the outer of $G \odot H$ is denoted by $V(H)$ and $E(H)$, respectively. We know that the corona product of any graph with a cycle produces a graph containing wheels. We have shown that the corona product of a cycle with either the complement of a complete graph, or a path, or a cycle is in $C_f 1$ [3] and the corona product of a complete graph with a cycle is in $C_f 1$ [1].

In this paper, we consider some other graphs containing wheels, namely the corona product of either the complement of a complete graph, or a path, or a star with a cycle. In Theorem 2.1, we give a classification of the corona product of the complement of a complete graph with a cycle. In [3], we have shown that the corona product of a cycle with a path is in $C_f 1$. It is well-known that the corona product of any two graphs is not commutative. Hence, it is natural to look for a sufficient condition of the corona product of a path with a cycle to be in $C_f 1$. In Theorem 2.2, we give a classification of the corona product of a path with a cycle. Moreover, in Theorem 2.3, we give a classification of the corona product of a star with a cycle.

2. Main Results

In this paper, we associate positive integers with colors. Let F be an f-coloring of G. Let $F^{-1}(i)$ denotes the set of edges of G that receive color i under F and $F_v^{-1}(i)$ denote the set of edges of G which is incident with the vertex v and receive color i under F.

Let $W_m = K_1 \odot C_m$ be a wheel on $m + 1$ vertices. Let $E_0 = E(C_m)$ and E_1 be a set of edges which are incident with $V(K_1)$. Let C be an f-coloring of W_m such that for $i \in [1, k]$, $|C_v^{-1}(i) \cap E_0| = 1$ for every $v \in V(C_m)$ (except when $k = 2$ and $|E_0|$ is odd, there is one and only one vertex $v \in V(C_m)$ with $|C_v^{-1}(i) \cap E_0| = 2$) and $a_1 \geq a_2 \geq ... \geq a_k$ where $a_i = |C^{-1}(i) \cap E_0|$. Let $b_i = |C^{-1}(i) \cap E_1|$, we have two following conditions:

1. If m is even and $a_i = \frac{m}{2} - t_i$ for some $t_i \in [0, (\frac{m}{2} - 1)]$, then $b_i \leq 2t_i$,
2. If m is odd and $a_i = \lfloor \frac{m}{2} \rfloor - t_i$ for some $t_i \in [0, (\lfloor \frac{m}{2} \rfloor - 1)]$, then $b_i \leq 2t_i + 1$.

We will use the f-coloring C to prove Theorem 2.1, Theorem 2.3, and Theorem 2.3.

In the Theorem 2.1, we give a classification of the corona product of the complement of a complete graph (K_n^c) with a cycle based on f-colorings.

Theorem 2.1 *Let $n \geq 1$, $m \geq 3$ and $G = K_n^c \odot C_m$.*
If either $m = 5$ or $m = 8$ and $f(v) = \begin{cases} 1, & v \in V(C_m), \\ \lceil \frac{m}{3} \rceil, & v \in V(K_n^c), \end{cases}$
then $G \in C_f2$. Otherwise, $G \in C_f1$.

Proof.

Let $n = 1$. If f fulfills the premise of the theorem, then $\Delta_f(G) = 3$. If $\Delta_f(G) = 1$, it is clear that we can construct an f-coloring of G with one color. For $\Delta_f(G) \geq 2$, we divide the proof into three cases as follows.

Case 1. $\Delta_f(G) = 2$
We can color all edges in E_0 and E_1 such that $|C_v^{-1}(i)| \leq 2$ for every $v \in V(C_m)$ and $|C_w^{-1}(i)| \leq \lceil \frac{m}{2} \rceil$ for the vertex $w \in V(K_1)$. Hence, $G \in C_f1$.

Case 2. $\Delta_f(G) = 3$
In this case, we divide the proof into three subcases.

Subcase 2.1 $m \neq 5$ and $m \neq 8$. Let $m = 3r + s$ for some $s \in [0, 2]$ and $r \in \mathbb{N}$. We can construct an f-coloring C by using 3 colors such that $|C_v^{-1}(i)| = 1$ for every vertex $v \in V(C_m)$ and $|C_w^{-1}(i)| \leq r + 1$ for the vertex $w \in V(K_1)$. Hence, $G \in C_f1$.

Subcase 2.2 For either $m = 5$ or $m = 8$ and f fulfills the premise of the theorem. We assume that there exists an f-coloring C of G by using 3 colors, we have the conditions as follows.
For $m = 5$, let h_1, h_2, h_3 be a monotone non-increasing sequence of the number of edges in E_0 with a same color such that $h_1 \geq h_2 \geq h_3$. Let $t_i \in [0, 1]$, since $h_1 + h_2 + h_3 = 5$, we get $t_1 + t_2 = 0$. Let g_1, g_2, g_3 be a number of edges in E_1 with a same color, respectively. Since $g_1 + g_2 + g_3 = 5$, we get $g_3 \geq 3$. It means that $f(w)$ must be 3 for $w \in V(K_1)$. We get a contradiction. It is impossible to construct an f-coloring by using 3 colors. Hence, $G \in C_f2$.
For $m = 8$, let $t_i \in [0, 3]$. By using a similar technique, we have $h_1 + h_2 + h_3 = 8$. it implies $t_1 + t_2 = h_3$. Since $g_1 + g_2 + g_3 = 8$, we get $2h_3 + g_3 \geq 8$ where $h_3 \leq 2$. It implies $h_3 \geq 4$. It means that $f(w)$ must be 4 for $w \in V(K_1)$.

We get a contradiction. It is impossible to construct an f-coloring by using 3 colors. Hence, $G \in C_f 2$.

Subcase 2.3 For either $m = 5$ or $m = 8$ and there exists $u \in V(C_m)$ with $f(u) \neq 1$ or $w \in V(K_1)$ with $f(w) \neq \lceil \frac{m}{3} \rceil$, we can color all edges of G by using the similar technique in Subcase 2.1 by using 3 colors such that $C_u^{-1}(i) \neq 1$ or $C_w^{-1}(i) \neq \lceil \frac{m}{3} \rceil$ for some $i \in [1,3]$, $u \in V(C_m)$ and $w \in V(K_1)$. Hence, $G \in C_f 1$.

Case 3. $\Delta_f(G) \geq 4$
We obtain $V^* \subseteq V(K_1)$. If $V_0^* = V(K_1)$, by Theorem 1.1, we have $G \in C_f 1$. Otherwise, by Theorem 1.2, we have $G \in C_f 1$.

Let $n \geq 2$. Since, $K_n^c \odot C_m$ is the n-copies of $K_1 \odot C_m$, we have a conclusion that if f fulfills the premise of the theorem, then $G \in C_f 2$. Otherwise, $G \in C_f 1$. □

Let $G = P_n \odot C_m$ be the corona product of a path on n vertices with a cycle on m vertices. Thus, we have the following theorem:

Theorem 2.2 Let $n \geq 2$, $m \geq 3$, and $G = P_n \odot C_m$.
If $m = 5$ and $f(v) = \begin{cases} 1, & \text{for } v \in V(C_m), \\ \lceil \frac{m}{3} \rceil, & \text{for } v \in V(P_n) \text{ and } d(v) = m+1, \end{cases}$
then $G \in C_f 2$. Otherwise, $G \in C_f 1$.

Proof.

We can color all edges of wheels by using the similar technique for every case in Theorem 2.1. Next, we color every edge of the path in three following cases.

Case 1 $\Delta_f(G) = 2$
We color all edges of the path by 1. Hence, $G \in C_f 1$.

Case 2 $\Delta_f(G) = 3$
We divide the proof into two subcases.

Subcase 1 For $m = 5$ and f fulfills the premise of the theorem, by using the same reason in the proof of Theorem 2.1 subcase 2.2, we can not construct an f-coloring by using 3 colors. Hence, $G \in C_f 2$.

Subcase 2 For $m \neq 5$ or $f(v)$ does not fulfill the premise of the theorem, we color all edges of the path by 1,2 alternately. But, when $m = 4$ or $m = 7$, we color all edges of the path by 1. Hence, $G \in C_f 1$.

Case 3 $\Delta_f(G) \geq 4$
We obtain $V^* \subseteq V(P_n)$. If there exists $v \in V(P_n)$ such that $v \in V_0^*$, by Theorem 1.1, we have $G \in C_f1$. Otherwise, by Theorem 1.2, we have $G \in C_f1$.
□

Let S_n be a star on $n+1$ vertices. In Theorem 2.3, we give a classification of the corona product of a star with a cycle based on f-colorings.

Theorem 2.3 Let $n \geq 3$ and $m \geq 3$, $G = S_n \odot C_m$.
If $m = 5$ and $f(v) = \begin{cases} 1, & v \in V(C_m), \\ \lceil \frac{m}{3} \rceil, & v \text{ are } n \text{ pendant vertices in } V(S_n), \end{cases}$
then $G \in C_f2$. Otherwise, $G \in C_f1$.

Proof.

Suppose, we can construct an f-coloring C by using $\Delta_f(G)$ colors. Let the center of the star be labeled by w and are pendant vertices of the star be labeled by $v_1, v_2, ..., v_n$, respectively. If f fulfills the premise of the theorem, then $\Delta_f(G) = 3$. If $\Delta_f(G) = 1$, it is clear that an f-coloring of G with one color. For $\Delta_f(G) \geq 2$, we divide the proof into three cases as follows.

Case 1. $\Delta_f(G) = 2$
For $i \in [1, \lceil \frac{m}{2} \rceil]$, we color all edges of wheels which the center of wheel is v_{2i-1} by using f-coloring C such that $|C_v^{-1}(i)| \leq 2$ for every $v \in V(C_m)$ and $|C_{v_{2i-1}}^{-1}(i)| \leq \lceil \frac{m}{2} \rceil$. If the center of wheel is v_{2i}, we color all edges of wheels by using the similar technique but we have to replace color 1 with 2 and otherwise. Finally, for $i \in [1, n]$, we color wv_i by 1, 2 alternately and we color the wheel which the center of wheel is w by using f-coloring C. Hence, $G \in C_f1$.

Case 2. $\Delta_f(G) = 3$
We divide the proof into three subcases.

Subcase 2.1 For $m \neq 5$. Let $m = 3r + s$ for some $s \in [0, 2]$ and $r \in [1, \lceil \frac{m}{3} \rceil)]$. We can color all edges of wheels as follows.

1. If the center of the wheel is v_{3r-2}, then we construct an f-coloring C by using the similar technique in the Theorem 2.1 Subcase 2.1.
2. If the center of the wheel is v_{3r-1}, then we construct an f-coloring C by using the similar technique in the Theorem 2.1 Subcase 2.1, but we have to replace color 1 with 2 and otherwise.
3. If the center of the wheel is v_{3r}, then we construct an f-coloring C by using the similar technique in the Theorem 2.1 Subcase 2.1, but we have to replace color 1 with 3 and otherwise.

Finally, for all edges of the wheel which the center of the wheel is w, we construct an f-coloring by using the similar technique in the Theorem 2.1 Subcase 2.1 and for $j \in [1, m]$, we color wv_j by $1, 2, 3$, alternately. Hence, $G \in C_f1$.

Subcase 2.2 For $m = 5$ and f fulfills premise of the theorem, by using the same reason in the proof of the Theorem 2.1 Subcase 2.2, we can not construct an f-coloring by using 3 colors. Hence, $G \in C_f2$.

Subcase 2.3 For $m = 5$ and there exists $u \in V(C_m)$ with $f(u) \neq 1$ or $w \in V(K_1)$ with $f(w) \neq \lceil \frac{m}{3} \rceil$, we can color all edges of G by using the similar technique in subcase 2.1 by using 3 colors such that $C_u^{-1}(i) \neq 1$ or $C_v^{-1}(i) \neq \lceil \frac{m}{3} \rceil$ for some $i \in [1, 3]$, $u \in V(C_m)$ or $v \in \{v_1, v_2, ..., v_n\}$. Hence, $G \in C_f1$.

Case 3. $\Delta_f(G) \geq 4$
We obtain $V^* \subseteq V(S_n)$. If there exist $v \in V(S_n)$ such that $v \in V_0^*$, by Theorem 1.1, we have $G \in C_f1$. Otherwise, by Theorem 1.2, we have $G \in C_f1$. □

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