Basic Notions on Graphs

Terminology

A graph G consists of a set V of vertices and a set E of edges which is a subset of $V \times V$.

Order: the number of vertices in the graph

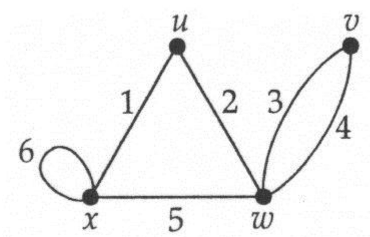
Degree: the number of edges attached to a vertex

- in a graph we have **maximum degree**, **minimum degree**
- if every vertex has the same degree then G is **regular.**

➔ 2 or more edges joining the same pair of vertices are called ***multiple edges***. An edge joining a vertex to itself is called a ***loop***.

Adjacency and incidence

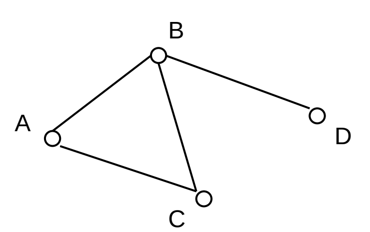
Two vertices *v* and *w* are ***adjacent*** vertices if they are joined by an edge *e*. The vertices *v* and *w* are ***incident*** with the edge *e,* and the edge *e* is ***incident*** with the vertices *v* and *w*.



Example

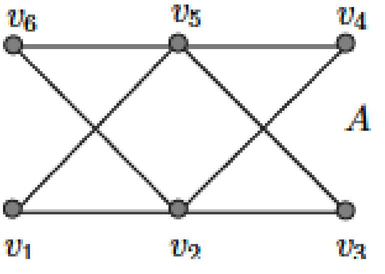
- G = G(V,E)
- V = {A,B,C,D}--The vertex set.
- E = {(A,B), (A,C), (B,C), (B,D)}--The edge set.

	A	B	C	D
A		1	1	
B	1		1	1
C	1	1		
D		1		



Terminology

Neighbourhood of a vertex is the set of all its adjacent vertices



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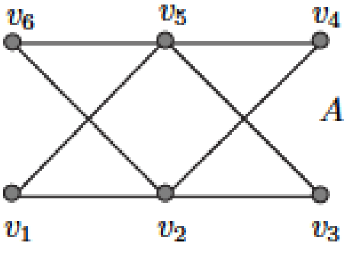
Terminology

Adjacency matrix A



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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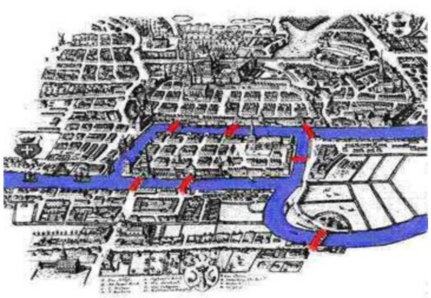
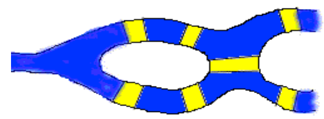
The Bridges of Konigsberg





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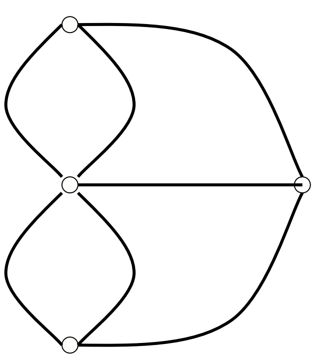
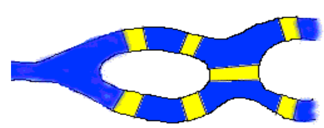
The Bridges of Konigsberg





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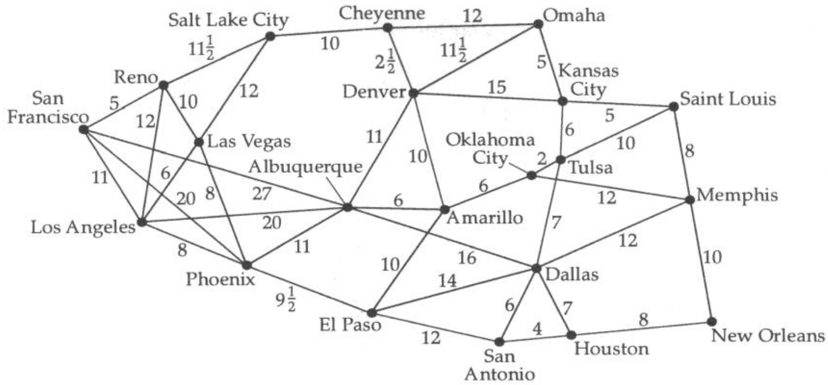
Weighted Graphs



Problem Find the shortest time taken to drive from Los Angeles to Amarillo, and from San Francisco to Denver.

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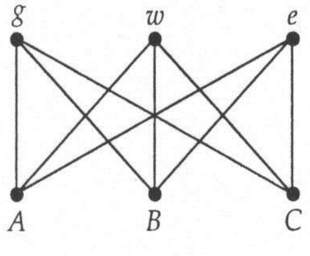
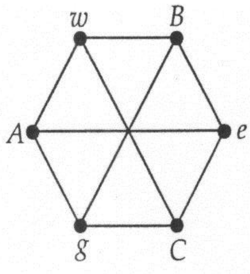
Isomorphism

Two graphs are the ***same*** if they have the same set of vertices and the same set of edges, even if they are drawn differently.





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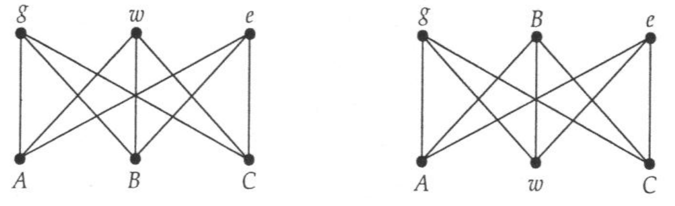
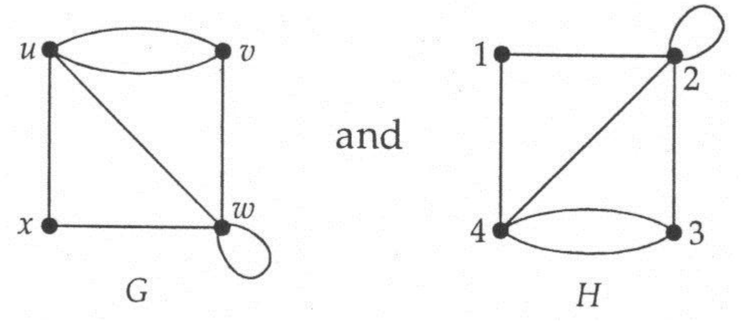
Isomorphism

Two graphs *G* and *H* are the ***isomorphic*** if *H* can be obtained by relabelling the vertices of *G*. That is, there is a 1-1 correspondence between the vertices of *G* and those of *H*, such that the number of edges joining each pair of vertices in *G* is equal to the number of edges joining the corresponding pair of vertices in *H*. Such a 1-1 correspondence is called an ***isomorphism***.



Isomorphism

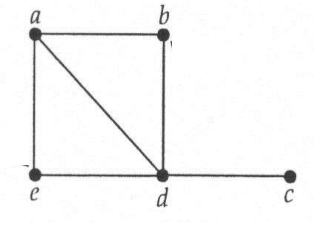


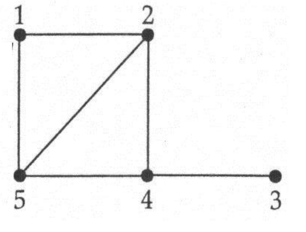
G and *H* are not the same but they are isomorphic: mapping *u* to 4; *v* to 3; *w* to 2; *x* to 1.
(*Check that the edges also correspond*!)

Isomorphism

Checking if two graphs are isomorphic:

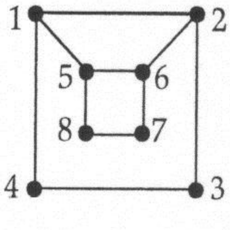
- number of vertices and edges
- look for special features such as short cycles degrees of vertices, loops, or multiple edges

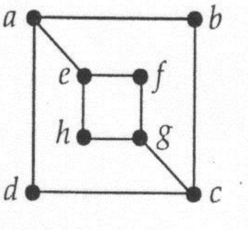




Isomorphism

Problem Are the following two graphs isomorphic? If so, find a suitable 1-1 correspondence between the vertices of the first and those of the second graph; if not, explain why not.

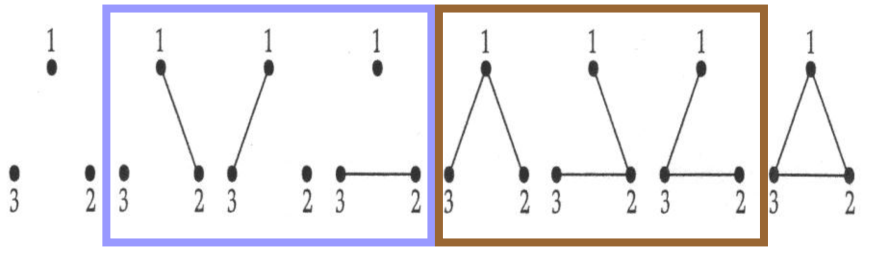




Counting graphs

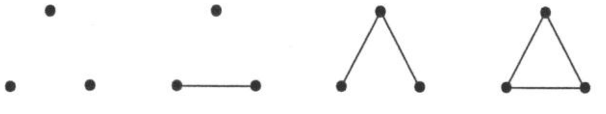
How many labelled and unlabelled graphs with the same number of vertices are there?

When counting *labelled graphs*, we distinguish between any two that are *not the same*.



Counting graphs

When counting *unlabelled graphs*, we distinguish between any two that are *not isomorphic*.



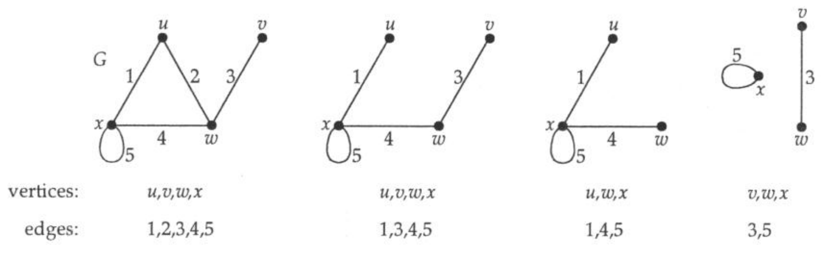
n	1	2	3
labelled graphs	1	2	8
unlabelled graphs	1	2	4

How many unlabelled, connected graphs are there on 1, 2, 3, 4, 5 vertices?

Subgraphs

A ***subgraph*** of a graph *G* is a graph all of whose vertices are vertices of *G* and all of whose edges are edges of *G*.

Example of a graphs and some of its subgraphs:



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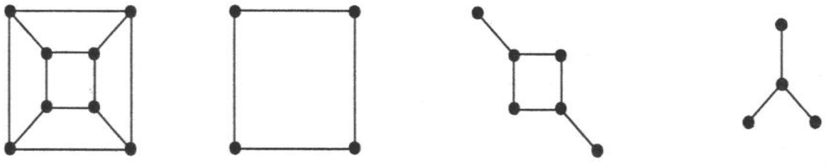
Subgraphs of unlabelled graphs

The idea of a subgraph can be extended also to unlabelled graphs:



H (a) (b) (c)

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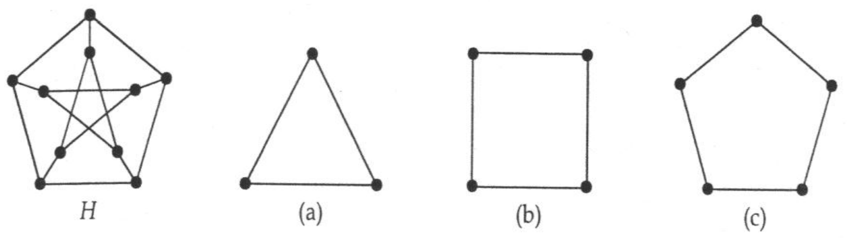
Subgraphs of unlabelled graphs

Problem Which of the following graphs are subgraphs of the graph *H* below?



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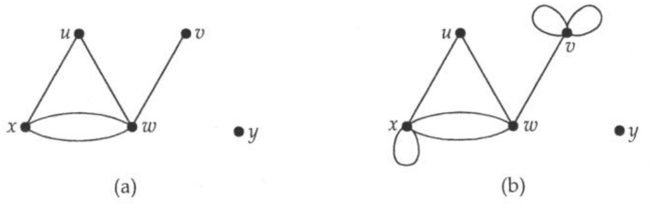
Vertex degrees



Graph (a) has vertex degrees
deg u=2, deg v=1, deg w=4, deg x=3, deg y=0
Graph (b) has vertex degrees
deg u=2, deg v=5, deg w=4, deg x=5, deg y=0

The ***degree sequence*** of a graph *G* is the sequence obtained by listing the vertex degrees of *G* in increasing order, with repeats as necessary.

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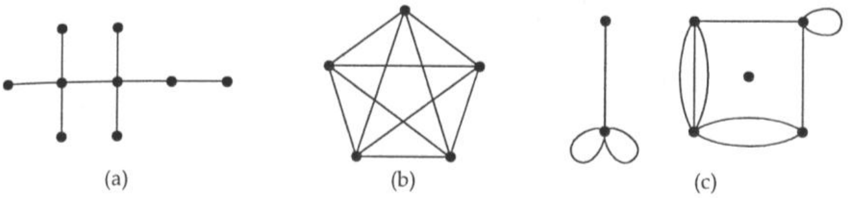
Degree sequences

Problem Write down the degree sequence of each of the following graphs.



(a) (b) (c)

Drawing a graph

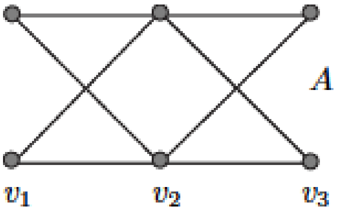
Degree Sequence:

- 4 4 4 4 4
- 6 6 6 6 4 3 3 0
- 5 4 3 2 2 1
- 6 5 5 4 3 3 2 2 2

Terminology

Walk of length *t* is a non-empty alternating sequence of *t* edges such that any two consecutive edges share a vertex

Diameter is the longest distance between any two vertices in the graph

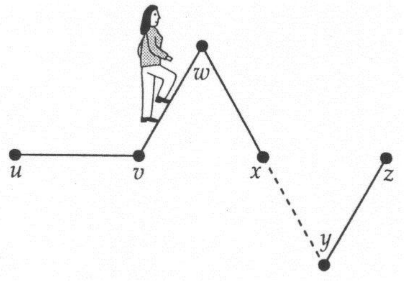


Walks

A **walk of length *k*** in a graph is a succession of *k* edges of the form

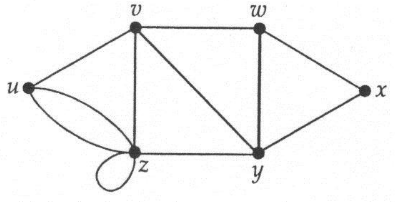
uv, vw, wx, …, yz.

This walk is denoted by *uvwx…yz*, and is referred to as a **walk between *u* and *z***.



Walks

Note that in a walk, we do not require the vertices and edges to be all distinct. For example, below *uvwxywvzzy* is a walk of length 9 between *u* and *y*; it includes the edge *vw* twice and the vertices *v,w,y* and *z* each twice.



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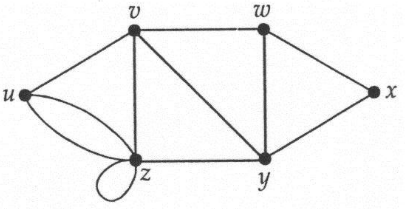
Paths and trails

A ***trail*** is a walk in *G* with the property that no edge is repeated.

A ***path*** is a trail in *G* with the property that no vertex is repeated..



Problem Complete the following statements:
(a) *xyzzvy* is a…….of length……between….and….;
(b) *uvyz* is a…….of length……between….and…..

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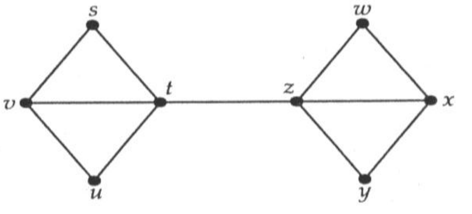
Paths

Problem Write down all the paths between s and y in the following graph:



Closed walks, trails and cycles

A ***closed walk*** is a succession of edges of the form

uv,vw,wx,…,yz,zu

That starts and ends at the same vertex.

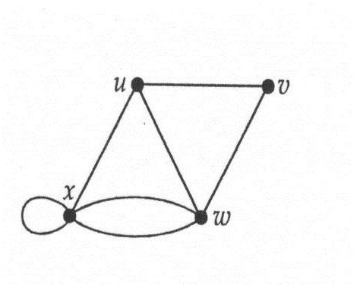
A ***closed trail*** ➔ a ***circuit***

A ***closed path*** ➔ a ***cycle***

➔ ***Eulerian Circuit (degree of every vertex of G is even)***
➔ ***Hamiltonian Cycle (a cycle that contains every vertex of G)***

Closed walks, trails and cycles

Problem For this graph, write down
(a) closed walk that is not a closed trail;
(b) closed trail that is not a cycle;
(c) all the cycles of lengths 1,2,3 and 4.



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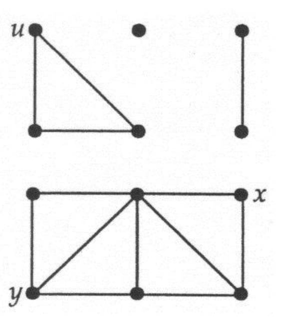
Connected graphs

A graph is ***connected*** if there is a path between each pair of vertices, and is ***disconnected*** otherwise.



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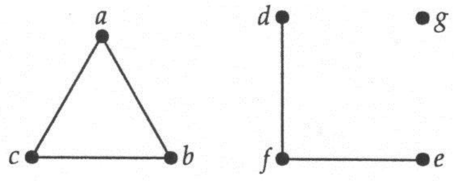
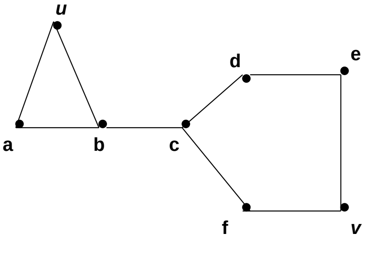
Components

An edge in a connected graph is a ***bridge*** if its removal results in a disconnected graph. Every disconnected graph consists of a number of connected subgraphs, called ***components***.



Distances in Graphs

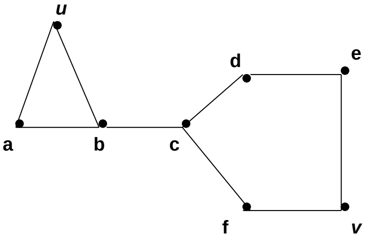
- The distance between two vertices *u* and *v* in a graph d(*u*,*v*) is the length of the shortest path between the two vertices.
- That is; the fewest number of edges that need to be traversed when going from *u* to *v*.



The distance from *u* to *v* is 4 by the path *u*bcf*v*. There are other paths from *u* to *v* but none shorter than 4.

Distances in Graphs

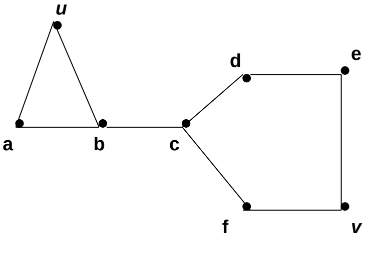
- The maximum distance from a vertex *u* to any other vertex in the graph is called the eccentricity of *u* ➔e(*u*).



e(u) = 4
e(b) = 3
e(e) = ?
e(a) = ?

Distances in Graphs

- The maximum distance from a vertex *u* to any other vertex in the graph is called the eccentricity of *u* e(*u*).
- The largest eccentricity is called the diameter, the smallest eccentricity is called the radius.

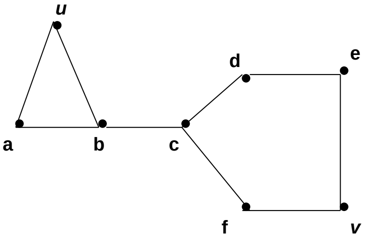


e(u) = 4
e(b) = 3
e(e) = 4
e(a) = 4

Note: If the graph is disconnected, the radius and diameter are infinity

Distances in Graphs

- The maximum distance from a vertex *u* to any other vertex in the graph is called the **eccentricity** of *u* e(*u*).
- The largest eccentricity is called the **diameter**, the smallest eccentricity is called the **radius**.



Diameter = 4
Radius = 2

Diameter? P_n , C_n, W_n, K_n

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